PAUL WACHIRA
wachipaul@gmail.com
+254 702909635

Course content

- Functions: definition, domain, range, codomain, composition (or composite), inverse.
- Limits, continuity and differentiability of a function.
- Differentiation by first principle and by rule for x^n (integral and fractional n).
- Other techniques of differentiation, i.e., sums, products, quotients, chain rule; their applications to algebraic, trigonometric, logarithmic, exponential, and inverse trigonometric functions all of a single variable.
- Implicit and parametric differentiation.
- Applications of differentiation to: rates of change, small changes, stationary points, equations of tangents and normal lines, kinematics, and economics and financial models (cost, revenue and profit).
- Introduction to integration and its applications to area and volume.

References

[1] Calculus: Early Transcendentals (8th Edition) by James Stewart

[2] Calculus with Analytic Geometry by Roland E. Larson, Robert P. Hostetler and Bruce H. Edwards; 5th edition

[3] Calculus and Analytical Geometry (9th edition) by George B. Thomas and Ross L. Finney

[4] Advanced Engineering Mathematics (10th ed.) by Erwin Kreyszig

[5] Calculus by Larson Hostellem

LECTURE 1

1 Functions

To understand the word function, we consider the following scenario and definitions. For example, the growth of a sidling is an instance of a functional relation, since the growth may be affected by variations in temperature, moisture, sunlight, etc. If all these factors remain constant, then the *growth is a function of time.*

Definition 1.1 (Variables)**.** *A variable is an object, event, time period, or any other type of category you are trying to measure.*

Consider the formula used for calculating the volume of a sphere of radius r.

$$V = \frac{4}{3}\pi r^3 \tag{1}$$

Then,

i) V and r vary with different spheres. Hence, they are called variables.

ii) π and $\frac{4}{3}$ are constants, irrespective of the size of the sphere.

There are two types of variables, i.e., independent and dependent variables.

Definition 1.2 (Independent and dependent variables)**.** *Independent variable refers to the input value while dependent variable refers to the output value.*

For example from formula (1), the volume, V, depends on the value of the radius, r, of the sphere. In this case, r is called the independent variable while V is called the dependent variable since it is affected by the variation of r. Similarly, for the function $y = ax^2 + bx + c$, a, b and c are constants, x is the independent variable and y is the dependent variable.

Definition 1.3 (Function)**.** *A function is a rule that assigns/associates each element in the independent set, say X, to a unique element in the dependent set, say Y.*

Examples of functions are

i) Linear functions e.g., $y = x + 5$

ii) Quadratic functions e.g., $y = x^2 - 2x + 5$

iii) Cubic functions e.g., $y = x^3 - 1$

iv) Quartic functions e.g., $y = 2x^4 + x^3 - 1$

v) Trigonometric functions e.g., $y = \sin(2x + 5)$

vi) Logarithmic functions (log to base 10) e.g., $y = \log(3x + 1)$

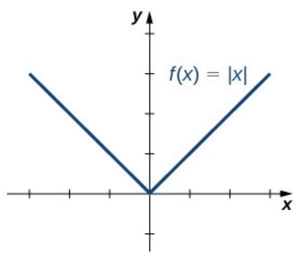
vii) Natural logarithmic functions (log to base $e \approx 2.71828$) e.g., $y = \ln(5x + 1)$

viii) Inverse of trigonometric functions e.g., $y = \tan^{-1}(2x + 1)$

ix) Exponential functions e.g., $y = e^{2x+1}$

x) Absolute value functions e.g., $y = |x|$. This function is defined as

$$y = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$



$\rightarrow$ Note: in the above examples the variable y depends on the variable x. Thus, we say that the dependent variable y is a function of the independent variable x. Using function notation, we write $y = f(x)$, where f is a function. The function $f(x)$ is read as f of x, meaning that f depends on x.

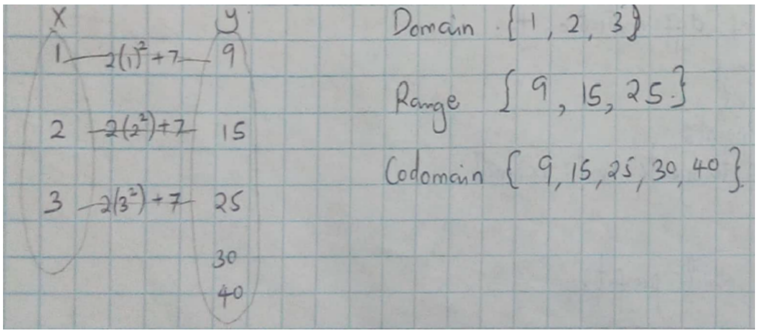
1.1 Domain, Range and Codomain

Definition 1.4 (Domain)**.** *A domain consists of all the elements in the independent set (i.e., the set of inputs), X, for which the function is defined.*

Definition 1.5 (Range)**.** *A range refers to a set of all the images of the elements in the domain.*

Definition 1.6 (Codomain)**.** *A codomain consists of all the elements in the dependent set (i.e., the set of outputs), Y.*

For example, consider the diagram below



Example(s):

1. Find the domain and range of the following functions.

   (a) $f(x) = (x-4)^2 + 5$

   Solution

   ☐ Since $f(x)$ is defined (or is a real number) for any real number x, the domain of f is the interval $(-\infty, \infty)$.

   ☐ Let $y = (x-4)^2 + 5$. Making x the subject, we have $x = 4 \pm \sqrt{y-5}$. This function is defined if $y - 5 \geq 0$ or $y \geq 5$. Therefore, the range is the interval $[5, \infty)$.

   (b) $f(x) = 2x^2 - 5x + 1$

   Solution

   ☐ Since $f(x)$ is defined (or is a real number) for any real number x, the domain of f is the interval $(-\infty, \infty)$.

   ☐ Let $y = 2x^2 - 5x + 1$ or $2x^2 - 5x + (1-y) = 0$. Making x the subject (use quadratic formula), we have $x = \dfrac{5 \pm \sqrt{25 - 8(1-y)}}{4}$. This function is defined if $25 - 8(1-y) \geq 0$ or $y \geq -\dfrac{17}{8}$. Therefore, the range is the interval $\left[-\frac{17}{8}, \infty\right)$.

   (c) $f(x) = \dfrac{4}{x^2 - 5x + 6}$

   Solution

   → Note: $4/0 = \infty$ (infinity), vvvv large value, undefined, indeterminate.

   ☐ The function $f(x)$ is defined when the denominator is nonzero, i.e., if $x^2 - 5x + 6 \neq 0$. Solving yields $x \neq 2$ and $x \neq 3$. Therefore, the domain of f includes all the real numbers of x except $x = 2$ and $x = 3$, i.e., the set $(-\infty, \infty)\backslash\{2, 3\}$ or $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

   ☐ Let $y = \dfrac{4}{x^2 - 5x + 6}$ or $x^2 - 5x + \left(6 - \frac{4}{y}\right) = 0$. Making x the subject (use quadratic formula), we have

   $$x = \frac{5 \pm \sqrt{25 - 4\left(6 - \frac{4}{y}\right)}}{2}$$

   This function is defined if $25 - 4\left(6 - \frac{4}{y}\right) \geq 0$ or $y \geq -16$. Therefore, the range is the interval $[-16, \infty)$.

(d) $f(x) = \sqrt{x-1}$

Solution

- ☐ Since $f(x)$ is defined (or is a real number) if $x - 1 \geq 0$ or $x \geq 1$, the domain of f is the interval $[1, \infty)$.
- ☐ Let $y = \sqrt{x-1}$. Making x the subject, we have $x = y^2 + 1$. This function is defined for any real number y. Therefore, the range is the interval $(-\infty, \infty)$.

(e) $f(x) = 2|x-3| + 4$

Solution

- ☐ Since $f(x)$ is defined for all real numbers, the domain of f is the interval $(-\infty, \infty)$.
- ☐ Since for all $|x-3| \geq 0$, the function $f(x) = 2|x-3| + 4 \geq 4$. Therefore, the range is all the values of y for which $y \geq 4$ or the interval $[4, \infty)$.

Exercise:

1. Find the domain and range of the following functions.

   (a) $f(x) = 6 - x^2$. [ans: domain $(-\infty, \infty)$, range $(-\infty, 6]$]

   (b) $f(x) = \dfrac{6+3x}{1-2x}$. [ans: domain $(-\infty, 0.5) \cup (0.5, \infty)$, range $(-\infty, 1.5) \cup (1.5, \infty)$]

   (c) $f(x) = \dfrac{x+5}{x-2}$. [ans: domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 1) \cup (1, \infty)$]

   (d) $f(x) = \sqrt{4-2x} + 5$. [ans: domain $(-\infty, 2]$, range $(-\infty, \infty)$]

   (e) $f(x) = \sqrt{\dfrac{x^2-16}{x^2-2x-24}}$. [ans: domain $(-\infty, -4) \cup [4, 6) \cup (6, \infty)$, range $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right] \backslash \{-1, 1\}$]

1.2 Evaluation of functions

This involves replacing x in the function by the suggested value and retaining the rule of the function.

Example(s):

1. Given $f(x) = 2x + 1$. Find: (i) $f(0)$, (ii) $f(1)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

   Solution

   i) $f(0) = 2(0) + 1 = 0 + 1 = 1$

   ii) $f(1) = 2(1) + 1 = 2 + 1 = 3$

   iii) $f(x+2) = 2(x+2) + 1 = 2x + 4 + 1 = 2x + 5$

   iv) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{[2(x+h)+1] - [2x+1]}{h} = \dfrac{2x + 2h + 1 - 2x - 1}{h} = \dfrac{2h}{h} = 2.$

2. Given $f(x) = 3x^2 - 2x + 4$. Find: (i) $f(0)$, (ii) $f(-1)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

   Solution

   i) $f(0) = 3(0)^2 - 2(0) + 4 = 0 + 0 + 4 = 4$

   ii) $f(-1) = 3(-1)^2 - 2(-1) + 4 = 3 + 2 + 4 = 9$

   iii) $f(x+2) = 3(x+2)^2 - 2(x+2) + 4 = 3(x^2 + 4x + 4) - 2x - 4 + 4 = 3x^2 + 10x + 12$

iv)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[3(x+h)^2-2(x+h)+4]-[3x^2-2x+4]}{h}\\ &= \frac{(3x^2+6hx+3h^2-2x-2h+4)-(3x^2-2x+4)}{h} = \frac{6hx+3h^2-2h}{h}\\ &= 6x+3h-2\end{aligned}$$

3. Given $f(x) = x^2 - 4x + 3$. Find: (i) $f(1)$, (ii) $f(2)$, (iii) $f(a)$, and (iv) $f(a+h)$.

Solution

i) $f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f(1) = 1^2 - 4(1) + 3 = 0$

ii) $f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f(2) = 2^2 - 4(2) + 3 = -1$

iii) $f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f(a) = a^2 - 4a + 3$

iv) $f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f(a+h) = (a+h)^2 - 4(a+h) + 3$

4. Given $\phi(\theta) = 2\sin\theta$. Find: (i) $\phi(\frac{\pi}{2})$, (ii) $\phi(0)$, and (iii) $\phi(\frac{\pi}{3})$.

Solution

i) $\phi(\theta) = 2\sin\theta \quad \Rightarrow \quad \phi(\frac{\pi}{2}) = 2\sin\left(\frac{\pi}{2}\right) = 2$

ii) $\phi(\theta) = 2\sin\theta \quad \Rightarrow \quad \phi(0) = 2\sin(0) = 0$

iii) $\phi(\theta) = 2\sin\theta \quad \Rightarrow \quad \phi(\frac{\pi}{3}) = 2\sin\left(\frac{\pi}{3}\right) = 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3}$

Exercise:

(a) Given $f(x) = x^3 + 2x + 1$, find: (i) $f(0)$, (ii) $f(-a)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h)-f(x)}{h}$ for $h \neq 0$.

(b) Given $g(x) = \dfrac{1}{\sqrt{x}+1}$, find: (i) $f(0)$, (ii) $f(1)$, (iii) $f(x+2)$, and (iv) $\dfrac{g(x+h)-g(x)}{h}$ for $h \neq 0$.

(c) Given $p(x) = \dfrac{6-2x}{1+3x}$, find: (i) $f(0)$, (ii) $f(-1)$, (iii) $f(2-x)$, and (iv) $\dfrac{p(x+h)-p(x)}{h}$ for $h \neq 0$.

(d) If $f(x) = 2x^2 - 4x + 1$, find (i) $f(1)$, (ii) $f(0)$, (iii) $f(2)$, (iv) $f(a)$, and $f(x+h)$.

(e) If $f(x) = (x-1)(x+5)$, find (i) $f(1)$, (ii) $f(0)$, (iii) $f(2)$, (iv) $f(a+1)$, and $f(\frac{1}{a})$.

(f) If $f(\theta) = \cos\theta$, find (i) $f(\frac{\pi}{2})$, (ii) $f(0)$, (iii) $f(\frac{\pi}{3})$, (iv) $f(\frac{\pi}{6})$, and (v) $f(\pi)$.

(g) If $f(x) = x^2$, find (i) $f(3)$, (ii) $f(3.1)$, (iii) $f(3.01)$, (iv) $f(3.001)$, and $\dfrac{f(3.001)-f(3)}{0.001}$.

(h) If $\phi(x) = 2^x$, find (i) $\phi(0)$, (ii) $\phi(1)$, and (iii) $\phi(0.5)$.

1.3 Composite functions

The composition of functions is a function of another function. Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(gof)(x)$ is the function with domain A and range E such that

$$(gof)(x) = g(f(x))$$

For example, given $f(x) = 2x + 1$ and $g(x) = 5x - 3$. Then,

$$(gof)(x) = g(f(x)) = g(2x+1) = 5(2x+1) - 3 = 10x + 2$$

Similarly,

$$(fog)(x) = f(g(x)) = f(5x-3) = 2(5x-3)+1 = 10x-5$$

→ Note: $(fog)(x) \neq (gof)(x)$.

Exercise:

1. Given $f(x) = x^2 - 1$, $g(x) = x - 1$ and $h(x) = \sqrt{x}$. Find:
   (a) $(fog)(x)$
   (b) $(hog)(x)$
   (c) $(gog)(x)$
   (d) $(gohof)(x)$
2. Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$. Evaluate
   (a) $(fog)(4)$
   (b) $(gof)(-1/2)$
3. If $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find the domain of $(fog)(x)$. [ans: $x \geq -0.5$ or $(-\infty, -0.5]$]

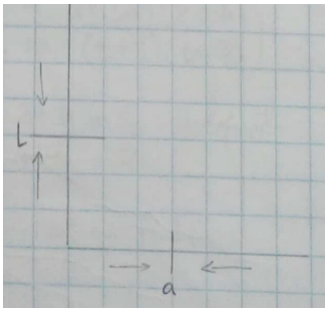
Lecture 2

2 Limits of functions

Definition 2.1 (Basic limit definition)**.** *Let $f(x)$ be a function and let a and L be real numbers. If $f(x)$ approaches L as x approaches a from either RHS or LHS of a (but is not equal to a), then we say that $f(x)$ has limit L as x approaches a, and is mathematically written as:*

$$\boxed{\lim_{x \to a} f(x) = L.}$$

Diagrammatically, we have



→ Note: $\lim_{x \to a} f(x)$ is the value that $f(x)$ approaches as x approaches a, and a does not have to be in the domain of f.

2.1 Properties of limits

Theorem 2.1. *Suppose $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. Then,*

1. [Addition/subtraction rule] $\lim_{x \to a} [f(x) \pm g(x)] = \left[\lim_{x \to a} f(x)\right] \pm \left[\lim_{x \to a} g(x)\right] = L_1 \pm L_2$
2. [Scalar multiple] $\lim_{x \to a} [\lambda f(x)] = \lambda \left[\lim_{x \to a} f(x)\right] = \lambda L_1$, where λ is a constant.
3. [Product rule] $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right] = L_1 \cdot L_2$
4. [Quotient rule] $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$, provided $g(a) \neq 0$.
5. $\lim_{x \to a} \sqrt[n]{f(x)} = \left(\lim_{x \to a} f(x)\right)^{1/n} = (L_1)^{1/n} = \sqrt[n]{L_1}$

→ Note: if $f(x) = c$ (where c is a constant), then $\lim_{x \to a} [f(x)] = \lim_{x \to a} [c] = c$

2.2 Techniques of evaluating limits of functions

☐ **Direct substitution (DS)**

The required limit is obtained by just plugging in the value of input, say x, into the given function, say $f(x)$.

Example(s):

(a) Evaluate $\lim\limits_{x\to 2} 3x^3 - x^2 + 2x + 5$.

Solution

$$\begin{aligned} \lim_{x\to 2}(3x^3 - x^2 + 2x + 5) &= 3\lim_{x\to 2} x^3 - \lim_{x\to 2} x^2 + 2\lim_{x\to 2} x + \lim_{x\to 2} 5) \\ &= 3(2^3) - (2^2) + 2(2) + 5 \\ &= 29 \end{aligned}$$

(b) Evaluate $\lim\limits_{x\to 1} \dfrac{x^2-1}{x+1}$.

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x+1} \overset{\text{D.S}}{=} \frac{1^2-1}{1+1} = \frac{0}{2} = 0$$

☐ **Factorization**

If on direct substitution we get the indeterminate form $0/0$, then it means that there is a common factor in both the numerator and denominator. In this case, we perform factorization first so as to simplify the given function.

→ Note: if the polynomial in the numerator is of degree greater than the degree of the polynomial in the denominator, we first need to perform long division.

Example(s):

(a) Evaluate $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

Solution

$$\begin{aligned} \lim_{x\to 2} \frac{x^2+x-6}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} \\ &= \lim_{x\to 2}(x+3) \overset{\text{D.S}}{=} 2+3 \\ &= 5 \end{aligned}$$

(b) Evaluate $\lim\limits_{x\to -2} \dfrac{x^2+3x+2}{2x^2-8}$

Solution

$$\begin{aligned} \lim_{x\to -2} \frac{x^2+3x+2}{2x^2-8} &= \lim_{x\to -2} \frac{(x+2)(x+1)}{2(x+2)(x-2)} \\ &= \lim_{x\to -2} \frac{x+1}{2(x-2)} \overset{\text{D.S}}{=} \frac{-2+1}{2(-2-2)} = \frac{-1}{-8} \\ &= \frac{1}{8} \end{aligned}$$

(c) Evaluate $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2-1}$.

Solution

$$\begin{aligned}
\lim_{x\to 1}\frac{x^3-1}{x^2-1} &= \lim_{x\to 1}\left(x+\frac{x-1}{x^2-1}\right) \quad \text{(long division)}\\
&= \lim_{x\to 1}\left[x+\frac{x-1}{(x-1)(x+1)}\right] \quad \text{(factorization)}\\
&= \lim_{x\to 1}\left(x+\frac{1}{x+1}\right) \overset{\text{D.S}}{=} 1+\frac{1}{1+1} = 1+\frac{1}{2}\\
&= \frac{3}{2}
\end{aligned}$$

☐ **Limits at infinity**

In this case, we first divide the numerator and denominator by the highest power of x in the denominator.

Example(s):

(a) Evaluate $\lim_{x\to\infty}\frac{5x^3-1}{4x^3-2x-7}$.

Solution

$$\begin{aligned}
\lim_{x\to\infty}\frac{5x^3-1}{4x^3-2x-7} &= \lim_{x\to\infty}\frac{5-\frac{1}{x^3}}{4-\frac{2}{x^2}-\frac{7}{x^3}} \overset{\text{D.S}}{=} \frac{5-\frac{1}{\infty}}{4-\frac{2}{\infty}-\frac{7}{\infty}} = \frac{5-0}{4-0-0}\\
&= \frac{5}{4}
\end{aligned}$$

☐ **Rationalization**

Suppose there exists a surd in either the numerator or denominator or both. Then, we first need to multiply both the numerator and denominator by the conjugate of the factor containing the surd (in either the numerator or denominator) and then simplify the resulting function. After rationalization, we perform a direct substitution.

→ Note: in case the surds appear in both the numerator and denominator, then we rationalize the denominator.

Example(s):

(a) Evaluate $\lim_{x\to\infty}\sqrt{x^2-4x}-x$.

Solution

$$\begin{aligned}
\lim_{x\to\infty}\sqrt{x^2-4x}-x &= \lim_{x\to\infty}\left(\sqrt{x^2-4x}-x\right)\left(\frac{\sqrt{x^2-4x}+x}{\sqrt{x^2-4x}+x}\right)\\
&= \lim_{x\to\infty}\frac{x^2-4x-x^2}{\sqrt{x^2-4x}+x} = \lim_{x\to\infty}\frac{-4x}{\sqrt{x^2-4x}+x}\\
&= \lim_{x\to\infty}\frac{-4x\cdot\frac{1}{x}}{\left(\sqrt{x^2-4x}+x\right)\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{-4}{\sqrt{1-\frac{4}{x}}+1}\\
&\overset{\text{D.S}}{=} \frac{-4}{\sqrt{1-\frac{4}{\infty}}+1} = \frac{-4}{\sqrt{1-0}+1} = \frac{-4}{1+1}\\
&= -2
\end{aligned}$$

(b) Evaluate $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

Solution

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} \\
&= \lim_{x\to 9} \frac{(x-9)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \\
&\overset{\text{D.S}}{=} \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} \\
&= \frac{1}{6}
\end{aligned}$$

Exercise:

(a) $\lim_{x\to\infty} \sqrt{x^2-2} - \sqrt{x^2+x}$.

Solution

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2-2} - \sqrt{x^2+x} &= \lim_{x\to\infty} \left(\sqrt{x^2-2} - \sqrt{x^2+x}\right) \cdot \frac{\sqrt{x^2-2}+\sqrt{x^2+x}}{\sqrt{x^2-2}+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty} \frac{(\cancel{x^2}-2)-(\cancel{x^2}+x)}{\sqrt{x^2-2}+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty} \frac{-\frac{2}{x}-1}{\sqrt{1-\frac{2}{x^2}}+\sqrt{1+\frac{1}{x}}} \overset{\text{D.S}}{=} \frac{-\frac{2}{\infty}-1}{\sqrt{1-\frac{2}{\infty}}+\sqrt{1+\frac{1}{\infty}}} \\
&= -\frac{1}{2}
\end{aligned}$$

(b) Evaluate $\lim_{x\to 1} \frac{x^3-1}{x-1}$. [ans: 3]

(c) Evaluate $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$. [ans: 1/2]

(d) Evaluate $\lim_{x\to\infty} \frac{5x^2-3x+2}{10x^2-x+100}$. [ans: 1/2]

(e) Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$. [ans: 1]

→ Note: A function which grows arbitrarily large as x goes to positive or negative infinity is said to have an infinite limit. **Infinity is not a real number, so if a function has infinite limit, we say that the limit does not exist.**

LECTURE 3

Theorem 2.2 (Squeeze law (sandwich theorem)). *Suppose that $f(x) \le g(x) \le h(x)$ holds for all x around a, except possibly at $x = a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example(s):

1. Find $\lim_{x\to 0} x \sin\left(\frac{1}{x^2+x}\right)$.

   Solution

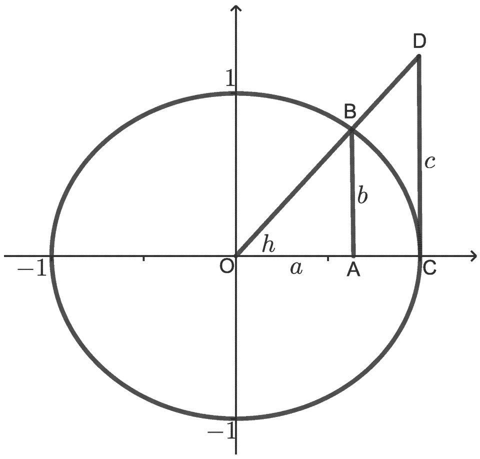
We know that $\sin\theta$ is sandwiched between -1 and 1 i.e., $-1 \leq \sin(\theta) \leq 1$. Therefore,

$$\begin{aligned}
\text{As} \quad & -1 \leq \sin\left(\frac{1}{x+x^2}\right) \leq 1 \\
\Rightarrow \quad & -x \leq x\sin\left(\frac{1}{x+x^2}\right) \leq x \\
\Rightarrow \quad & -\lim_{x\to 0}(x) \leq \lim_{x\to 0} x\sin\left(\frac{1}{x+x^2}\right) \leq \lim_{x\to 0}(x) \\
\Rightarrow \quad & 0 \leq \lim_{x\to 0} x\sin\left(\frac{1}{x+x^2}\right) \leq 0 \\
\Rightarrow \quad & \lim_{x\to 0} x\sin\left(\frac{1}{x+x^2}\right) = 0
\end{aligned}$$

Exercise:

1. [***Assignment 1***] Prove that $\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\displaystyle\lim_{h\to 0}\frac{(1-\cos h)}{h} = 0$.

   Proof. Consider the following unit circle. Let the length of line OA be a units, AB is b units, CD be c units and angle AOB be h.



Here, $OB = OC = 1$ unit. Now, $\cos h = a$, $\sin h = b$ and $\tan h = c$. From the figure, the area of triangle OAB is less than that of the sector OCB which is also less than that of triangle OCD i.e., $\frac{1}{2}ab \leq \frac{h}{2\pi}\cdot\pi(1)^2 \leq \frac{1}{2}(1)c$. Thus,

$$\frac{1}{2}\cos h\sin h \leq \frac{1}{2}h \leq \frac{1}{2}\tan h$$

Multiply through by 2 and using the identity $\tan h = \dfrac{\sin h}{\cos h}$, we have

$$\cos h\sin h \leq h \leq \frac{\sin h}{\cos h}$$

Taking reciprocals, we have

$$\frac{1}{\cos h\sin h} \geq \frac{1}{h} \geq \frac{\cos h}{\sin h}$$

Multiplying though by $\sin h$ yields $\dfrac{1}{\cos h} \geq \dfrac{\sin h}{h} \geq \cos h$, which can be rewritten as

$$\cos h \leq \frac{\sin h}{h} \leq \frac{1}{\cos h}$$

Taking limit as $h \to 0$, we have $\displaystyle\lim_{h\to 0}\cos h \leq \lim_{h\to 0}\frac{\sin h}{h} \leq \lim_{h\to 0}\frac{1}{\cos h}$. That is, $\displaystyle 1 \leq \lim_{h\to 0}\frac{\sin h}{h} \leq 1$. Hence, by the squeeze law we get

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h} = 1}$$

Also,

$$\begin{aligned}
\lim_{h\to 0}\frac{(1-\cos h)}{h} &= \lim_{h\to 0}\left[\frac{(1-\cos h)}{h}\cdot\frac{(1+\cos h)}{(1+\cos h)}\right] = \lim_{h\to 0}\left[\frac{1-\cos^2 h}{h}\cdot\frac{1}{1+\cos h}\right] \\
&= \lim_{h\to 0}\left[\frac{\sin^2 h}{h}\cdot\frac{1}{1+\cos h}\right] = \lim_{h\to 0}\left[\frac{\sin h}{h}\cdot\frac{\sin h}{1+\cos h}\right] \\
&= \left[\lim_{h\to 0}\frac{\sin h}{h}\right]\left[\lim_{h\to 0}\frac{\sin h}{(1+\cos h)}\right] = (1)\left[\lim_{h\to 0}\frac{\sin h}{(1+\cos h)}\right] \overset{D.S}{=} \frac{0}{(1+1)} \\
&= 0
\end{aligned}$$

Therefore,

$$\boxed{\lim_{h\to 0}\frac{(1-\cos h)}{h} = 0}$$

□

2.3 One-Sided Limit

Definition 2.2 (Left-Hand Limit)**.** *If a function $f(x)$ approaches the number L as x approaches the real number a from the LHS of a, then we say that L is the left-hand limit of f at $x = a$ and is written as:*

$$\boxed{\lim_{x\to a^-} f(x) = L.}$$

Definition 2.3 (Right-Hand Limit)**.** *If a function $f(x)$ approaches the number L as x approaches the real number a from the RHS of a, then we say that L is the right-hand limit of f at $x = a$ and is written as:*

$$\boxed{\lim_{x\to a^+} f(x) = L.}$$

→ Note: the limit of $f(x)$ as x approaches a exists if both left-hand limit and right-hand limit exist and are equal at $x = a$. In that case, we have

$$\boxed{\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = L}$$

Example(s):

(a) Consider the function defined by $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 2-x & \text{if } x > 1. \end{cases}$ Evaluate $\lim_{x\to 1} f(x)$.

Solution

(i) LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^3) = 1^3 = 1$

(ii) RHL: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x) = 2-1 = 1$

(iii) Since the result (i) = (ii), we get $\lim_{x\to 1} f(x) = 1$

Exercise:

(a) Consider the function defined by $f(x) = \begin{cases} x^2-2x & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x-4 & \text{if } x > 1. \end{cases}$. Evaluate $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Consider the function defined by $f(x) = \begin{cases} 2-3x & \text{if } x \le 1 \\ 2x^3 & \text{if } x > 1 \end{cases}$. Does $\lim_{x\to 1} f(x)$ exist?

(c) Find the value of $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x & \text{if } x \ne 0 \\ 1 & \text{if } x = 0. \end{cases}$

☐ Meaning of absolute value functions

To separate (or split) the function contained in the absolute value function, do the following:

i) First identify the reference point by equating the interior term to zero.

ii) Investigate the signs of the interior expression to the left and the right of the reference point.

For example,

☐ If $f(x) = |x-3|$. The reference point is $x - 3 = 0 \quad \Rightarrow \quad x = 3$. Thus,

$$f(x) = \begin{cases} -(x-3) & \text{if } x < 3 \\ +(x-3) & \text{if } x \geq 3 \end{cases}$$

☐ If $f(x) = 5 + |x+5|$. The reference point is $x + 5 = 0 \quad \Rightarrow \quad x = -5$. Thus,

$$f(x) = \begin{cases} 5-(x+5) & \text{if } x < -5 \\ 5+(x+5) & \text{if } x \geq -5 \end{cases}$$

☐ If $f(x) = \dfrac{1}{2-|x|}$. The reference point is $x = 0$. Thus, $f(x) = \begin{cases} \dfrac{1}{2+x} & \text{if } x < 0 \\ \dfrac{1}{2-x} & \text{if } x \geq 0 \end{cases}$.

Example(s):

(a) Evaluate $\lim\limits_{x\to 0} \dfrac{|5x|}{x}$.

Solution

The reference point is $5x = 0 \quad \Rightarrow \quad x = 0$. Thus, we have

$$f(x) = \frac{|5x|}{x} = \begin{cases} \frac{-(5x)}{x} & \text{if } x < 0 \\ \frac{+(5x)}{x} & \text{if } x > 0 \end{cases}$$

Now,

(i) LHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{-5x}{x} = -5$

(ii) RHL: $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{5x}{x} = 5$

(iii) Since (i) $\neq$ (ii), therefore, $\lim\limits_{x\to 0} \dfrac{|5x|}{x}$ does not exist. The above problem possesses a one-sided limits.

(b) Evaluate $\lim\limits_{x\to 7} \left(\dfrac{|x-7|}{(x-7)}\right)$.

Solution

The reference point is $x - 7 = 0 \quad \Rightarrow \quad x = 7$. Thus, we have

$$f(x) = \frac{|x-7|}{x-7} = \begin{cases} \frac{-(x-7)}{x-7} = -1 & \text{if } x < 7 \\ \frac{+(x-7)}{x-7} = +1 & \text{if } x > 7 \end{cases}$$

Now,

(i) LHL: $\lim\limits_{x\to 7^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$

(ii) RHL: $\lim\limits_{x\to 7^+} f(x) = \lim\limits_{x\to 0^+} (+1) = 1$

(iii) Since (i) $\neq$ (ii), therefore, $\lim_{x\to7} f(x)$ does not exist. The above problem possesses a one-sided limits.

Exercise:

(a) Evaluate $\lim_{x\to6}\left(\dfrac{x+6}{|x+6|}\right)$.

(b) Evaluate $\lim_{x\to1}\left(2+\dfrac{1}{|x-1|}\right)$.

3 Continuity of a function

A function $f(x)$ is said to be continuous at a point $x = a$ if the following three conditions are satisfied:

i) $f(a)$ is finite, i.e., $f(x)$ must be defined at $x = a$.

ii) $\lim_{x\to a} f(x)$ exists (i.e., LHL=RHL at $x = a$)

iii) $\lim_{x\to a} f(x) = f(a)$, i.e., (ii)=(i)

→ Note: if at least one of these conditions is not satisfied, then $f(x)$ is discontinuous at $x = a$. In this case, we say that the point a is a discontinuity of f (i.e., $f(x)$ has some gaps or jumps at $x = a$).

Example(s):

(a) Discuss the continuity of the function $f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if}\, x < -1 \\ x^2-3 & \text{if}\, x \geq -1 \end{cases}$ at $x = -1$

Solution

We need to test the three conditions for continuity:

(i) $f(-1) = (-1)^2 - 3 = -2$ (defined).

(ii) $\underline{\lim_{x\to-1} f(x)}$:

$$\text{LHL:} \quad \lim_{x\to-1^-} f(x) = \lim_{x\to-1^-}\left(\frac{x^2-1}{x+1}\right) = \lim_{x\to-1^-}\frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = \lim_{x\to-1^-}(x-1) = -2$$

$$\text{RHL:} \quad \lim_{x\to-1^+} f(x) = \lim_{x\to-1^+}(x^2-3) = 1-3 = -2$$

Since LHL=RHL=-2, therefore, $\lim_{x\to-1} f(x) = -2$

(iii) So, as $\lim_{x\to-1} f(x) = f(-1)$ therefore, $f(x)$ is continuous on $(-4, 4)$.

(b) Discuss the continuity of the function $f(x) = \dfrac{2x^4 - 6x^3 + x^2 + 3}{x-1}$ at $x = 1$.

Solution

Clearly, the function $f(x) = \dfrac{2x^4 - 6x^3 + x^2 + 3}{x-1}$ is discontinuous at $x = 1$. However, the point of discontinuity can be removed by first simplifying the given function. Thus, by long division we have

$$
\begin{array}{r|l}
 & 2x^3 - 4x^2 - 3x - 3 \\ \hline
x-1 & 2x^4 - 6x^3 + x^2 \qquad + 3 \\
 & -2x^4 + 2x^3 \\ \hline
 & \quad -4x^3 + x^2 \\
 & \quad 4x^3 - 4x^2 \\ \hline
 & \qquad -3x^2 \\
 & \qquad 3x^2 - 3x \\ \hline
 & \qquad\quad -3x + 3 \\
 & \qquad\quad 3x - 3 \\ \hline
 & \qquad\qquad 0
\end{array}
$$

Hence, the function can be rewritten in the simplest form $f(x) = 2x^3 - 4x^2 - 3x - 3$, which is now continuous at $x = 1$ [student to verify this]. Therefore, the original function is said to have a **removable point of discontinuity**.

(c) Find the value of the constants in the give problems if $f(x)$ is continuous everywhere in the real number line

i) Given $f(x) = \begin{cases} 4 + c & \text{if } x < 1 \\ 4x + 2 & \text{if } x \geq 1 \end{cases}$. Find c [ans: $c = 2$]

ii) Given $f(x) = \begin{cases} 15x & \text{if } x < -1 \\ ax + b & \text{if } -1 \leq x < 2 \\ 12x & \text{if } x > 2 \end{cases}$. Find a and b [ans: $a = 3, b = 18$]

Solution

In these questions, we make use of the second condition of continuity in particular, i.e., LHL=RHL at any point $x = a$.

i) The reference point is $x = 1$. Thus,

$$
\begin{array}{lrcl}
\text{LHL:} & \lim_{x \to 1^-} f(x) & = & \lim_{x \to 1^-} (4 + c) = 4 + c \\
\text{RHL:} & \lim_{x \to 1^+} f(x) & = & \lim_{x \to 1^+} (4x + 2) = 4 + 2 = 6
\end{array}
$$

Since $f(x)$ is continuous at $x = 1$, we have $4 + c = 6$. Therefore, $c = 2$.

ii) ☐ Case 1: Taking the reference point as $x = -1$. Thus,

$$
\begin{array}{lrcl}
\text{LHL:} & \lim_{x \to -1^-} f(x) & = & \lim_{x \to -1^-} (15) = 15 \\
\text{RHL:} & \lim_{x \to -1^+} f(x) & = & \lim_{x \to -1^+} (ax + b) = -a + b
\end{array}
$$

Since $f(x)$ is continuous at $x = -1$, we have $-a + b = 15 \qquad (*)$.

☐ Case 2: Taking the reference point as $x = 2$. Thus,

$$
\begin{array}{lrcl}
\text{LHL:} & \lim_{x \to 2^-} f(x) & = & \lim_{x \to 2^-} (ax + b) = 2a + b \\
\text{RHL:} & \lim_{x \to 2^+} f(x) & = & \lim_{x \to 2^+} (12x) = 24
\end{array}
$$

Since $f(x)$ is continuous at $x = 2$, we have $2a + b = 24 \qquad (**)$. Solving equations $(*)$ and $(**)$ simultaneously, we obtain $a = 3$ and $b = 18$.

Exercise:

(a) Discuss the continuity of the function $f(x) = \begin{cases} \dfrac{x^3+27}{x+3} & \text{if } x \neq -3 \\ 27 & \text{if } x = -3 \end{cases}$.

(b) Find the value of A and B so that the following function is continuous for all x.

$$f(x) = \begin{cases} A\left(\dfrac{1-\cos x}{\sin^2 x}\right) & \text{if } x < 0 \\ 2x^2 - x + B & \text{if } 0 \leq x \leq 1 \\ \dfrac{x^2+2x-3}{x^2-1} & \text{if } x > 1 \end{cases}$$

Solution

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{A(1-\cos(x))}{\sin^2(x)} = \lim_{x\to 0^-} \frac{A\cancel{(1-\cos(x))}}{\cancel{(1-\cos(x))}(1+\cos(x))} = \frac{A}{2} \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (2x^2 - x + B) = B \end{aligned}$$

Since $f(x)$ to be continuous at $x = 0$, we have $\dfrac{A}{2} = B - - - (*)$.

Also,

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (2x^2 - x + B) = 1 + B \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = \frac{4}{2} = 2 \end{aligned}$$

Since $f(x)$ to be continuous at $x = 1$, we have $1 + B = 2 - - - (**)$.

Solving equations $(*)$ and $(**)$, we get $A = 2, B = 1$.

(c) Find a and b so that the following functions are continuous $\forall x \in \mathbb{R}$:

i)

$$f(x) = \begin{cases} 2, & \text{if } x < 1 \\ ax + b, & \text{if } 1 \leq x < 2 \\ 6, & \text{if } x \geq 2 \end{cases}$$

[ans: $a = 4, b = -2$]

ii)

$$f(x) = \begin{cases} -2x, & \text{if } x < 1 \\ b - ax^2, & \text{if } 1 \leq x < 4 \\ -16x, & \text{if } x \geq 4 \end{cases}$$

[ans: $a =, b =$]

LECTURE 4

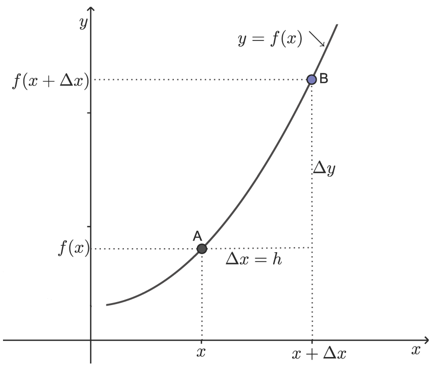
4 Derivative of functions

Definition 4.1 (First principle)**.** *The derivative of a function $f(x)$ denoted by $f'(x)$ or $\dfrac{df}{dx}$ is the rate of change of f with respect to x, and is given by*

$$\boxed{f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]},$$

for all x for which this limit exists.

The process of finding the derivative $f'(x)$ is called *differentiation* of $f(x)$. The above relation is called first principle of differentiation or differentiation by the definition or differentiation of first kind. Geometrically, consider the curve $y = f(x)$ and let $\Delta x = h$.



Here, $\Delta y = f(x+h) - f(x)$. So, gradient of the secant line through points A and B is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h)-f(x)}{h}$. Taking limit as $h \to 0$ yields

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Therefore,

$$\boxed{\frac{dy}{dx} = f'(x)}$$

Example(s):

(a) Use first principle of differentiation to find the derivative of the function $f(x) = x^2$.

Solution

Given $f(x) = x^2$, we have $f(x+h) = (x+h)^2$. By the first principle of differentiation, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0}\frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0}\frac{2hx+h^2}{h} = \lim_{h\to 0}(2x+h) \overset{\text{D.S}}{=} (2x+0) \\ &= 2x \end{aligned}$$

(b) Use first principle of differentiation to find the derivative of the following functions: (i) $f(x) = \dfrac{1}{x}$ and (ii) $f(x) = \sqrt{x}$.

Solution

i) Given $f(x) = \dfrac{1}{x}$, we have $f(x+h) = \dfrac{1}{x+h}$. By the first principle of differentiation, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} \\ &= \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0}\frac{-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} \overset{\text{D.S}}{=} \frac{-1}{x(x+0)} \\ &= -\frac{1}{x^2} \end{aligned}$$

ii) Given $f(x) = \sqrt{x}$, we have $f(x+h) = \sqrt{x+h}$. By the first principle of differentiation, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \overset{\text{D.S}}{=} \frac{1}{\sqrt{x+0}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Exercise:

(a) Use first principle of differentiation to find the derivative of the following functions.

i) $f(x) = -x^3 + 3x^2 + 4$ [ans: $f'(x) = -3x^2 + 6x$]

ii) $f(x) = \dfrac{3x}{1-5x}$ [ans: $f'(x) = \dfrac{3}{(1-5x)^2}$]

iii) $f(x) = \dfrac{-7+5x}{-3-2x}$ [ans: $f'(x) = \dfrac{-29}{(-3-2x)^2}$]

iv) $f(x) = \sqrt{6x+2} - 5$ [ans: $f'(x) = \dfrac{3}{\sqrt{6x+2}}$]

v) $f(x) = \dfrac{1}{\sqrt{x}+2}$ [ans: $f'(x) = \dfrac{-1}{2\sqrt{x}(\sqrt{x}+2)^2}$]

4.1 Basic differentiation rules

□ The derivative of a constant

If $f(x) = c$ (a constant) for all x, then $f'(x) = 0$ for all x. That is, $\boxed{\dfrac{dc}{dx} = f'(x) = 0}$.

Proof. Given $f(x) = c \quad \Rightarrow \quad f(x+h) = c$. Thus, from the first principle, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

□

□ The power rule

If $f(x) = x^n$ for $n \in \mathbb{R}$, then $\boxed{f'(x) = nx^{n-1}}$. That is, bring down the power and reduce the power by one.

Proof. Given $f(x) = x^n \quad \Rightarrow \quad f(x+h) = (x+h)^n = x^n + nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n$. Thus, from the first principle, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n - x^n\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n\right)}{h} \\ &= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + h^{n-1}\right) \\ &\overset{\text{D.S}}{=} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(0) + \cdots + (0)^{n-1}\right) = nx^{n-1} \end{aligned}$$

□

For example,

i) If $f(x) = 6x^5$, then $f'(x) = 30x^4$.
ii) If $f(x) = x^{10}$, then $f'(x) = 10x^9$.

☐ **The derivative of a linear combination**

If $f(x)$ and $g(x)$ are differentiable functions of x and a and b are constants, then

$$\boxed{\frac{d}{dx}\left[af(x) + bg(x)\right] = af'(x) + bg'(x)}$$

Proof. Let $y(x) = af(x) + bg(x)$. Thus, from the first principle, we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim_{h\to 0}\frac{[af(x+h)+bg(x+h)]-[af(x)+bg(x)]}{h} \\
&= \lim_{h\to 0}\frac{a[f(x+h)-f(x)]+b[g(x+h)-g(x)]}{h} \\
&= a\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + b\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= af'(x)+bg'(x)
\end{aligned}$$

☐

For example,

i) If $y = 24x + 8x^5$, then $\dfrac{dy}{dx} = 24 + 40x^4$.
ii) If $y = 7x^3 - 9x^2 + 4x + 2$, then $\dfrac{dy}{dx} = 21x^2 - 18x + 4$.

☐ **The product rule**

If $u(x)$ and $v(x)$ are differentiable functions of x, then the product $u(x)v(x)$ is also a differentiable function of x, and

$$\boxed{\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)}$$

Proof. Let $y(x) = u(x)v(x)$. Thus, from the first principle, we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h} \\
&= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h} \\
&= \lim_{h\to 0}\frac{[u(x+h)-u(x)]\,v(x+h)+u(x)\,[v(x+h)-v(x)]}{h} \\
&= \lim_{h\to 0}\frac{[u(x+h)-u(x)]\,v(x+h)}{h} + \lim_{h\to 0}\frac{u(x)\,[v(x+h)-v(x)]}{h} \\
&= \left[\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\right]\left[\lim_{h\to 0}v(x+h)\right] + u(x)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} \\
&= u'(x)v(x)+u(x)v'(x)
\end{aligned}$$

☐

→ Note: the product rule says that the derivative of the product of two functions is formed by multiplying the derivative of each function by the other function and then adding the results. In general, suppose $y = u_1(x)u_2(x)\cdots u_n(x)$, then

$$\boxed{\frac{dy}{dx} = u_1'(x)u_2(x)\cdots u_n(x) + u_1(x)u_2'(x)\cdots u_n(x) + \cdots + u_1(x)u_2(x)\cdots u_n'(x)}$$

Example(s):

(a) Find the derivative of $f(x) = (1 - 5x^2)(6x^2 - 4x + 1)$.

Solution
Let $f(x) = uv$, where $u = 1 - 5x^2$ and $v = 6x^2 - 4x + 1$. Differentiating yields $u' = -10x$ and $v' = 12x - 4$. Therefore,

$$\begin{aligned} f'(x) &= u'v + uv' = (-10x)(6x^2 - 4x + 1) + (1 - 5x^2)(12x - 4) \\ &= -60x^3 + 40x^2 - 10x + 12x - 4 - 60x^3 + 20x^2 \\ &= -120x^3 + 60x^2 + 2x - 4 \end{aligned}$$

(b) Find the derivative of $y = (x - 2)(x^2 + 6)(x^4 + 1)$.

Solution
Let $y = uvw$, where $u = x - 2$, $v = x^2 + 6$ and $w = x^4 + 1$. Differentiating yields $u' = 1, v' = 2x$ and $w' = 4x^3$. Therefore,

$$\begin{aligned} \frac{dy}{dx} &= u'vw + uv'w + uvw' \\ &= (1)(x^2 + 6)(x^4 + 1) + (x - 2)(2x)(x^4 + 1) + (x - 2)(x^2 + 6)(4x^3) \\ &= \left(x^6 + x^2 + 6x^4 + 6\right) + 2x\left(x^5 + x - 2x^4 + 2\right) + 4x^3\left(x^3 + 6x - 2x^2 - 12\right) \\ &= x^6 + x^2 + 6x^4 + 6 + 2x^6 + 2x^2 - 4x^5 + 4x + 4x^6 + 24x^4 - 8x^5 - 48x^3 \\ &= 7x^6 - 12x^5 + 30x^4 - 48x^3 + 3x^2 + 4x + 6 \end{aligned}$$

□ The quotient rule

If $u(x)$ and $v(x)$ are differentiable functions of x, then the quotient $\dfrac{u(x)}{v(x)}$ (where $v(x) \neq 0$) is also a differentiable function of x, and

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Proof. Let $y(x) = \dfrac{u(x)}{v(x)}$. Thus, from the first principle, we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} = \lim_{h\to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x)v(x+h)} \\ &= \lim_{h\to 0} \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x)v(x+h)} \\ &= \lim_{h\to 0} \frac{[u(x+h) - u(x)]\,v(x) - u(x)\,[v(x+h) - v(x)]}{hv(x)v(x+h)} \\ &= \frac{\lim\limits_{h\to 0} \dfrac{[u(x+h) - u(x)]}{h} v(x) - u(x) \lim\limits_{h\to 0} \dfrac{[v(x+h) - v(x)]}{h}}{\lim\limits_{h\to 0} v(x)v(x+h)} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

□

Example(s):

(a) Differentiate $y = \dfrac{2x^2+1}{x^2-1}$.

Solution
Let $y = \dfrac{u}{v}$, where $u = 2x^2+1$ and $v = x^2-1$. Differentiating yields $u' = 4x$ and $v' = 2x$. Therefore,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x^2-1)(4x) - (2x^2+1)(2x)}{(x^2-1)^2} = \frac{4x^3 - 4x - 4x^3 - 2x}{(x^2-1)^2} = \frac{-6x}{(x^2-1)^2}$$

(b) Differentiate $y = \dfrac{x^3}{x-1}$.

Solution
Let $y = \dfrac{u}{v}$, where $u = x^3$ and $v = x-1$. Differentiating yields $u' = 3x^2$ and $v' = 1$. Therefore,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x-1)(3x^2) - (x^3)(1)}{(x-1)^2} = \frac{3x^3 - 3x^2 - x^3}{(x-1)^2} = \frac{2x^3 - 3x^2}{(x-1)^2}$$

□ The chain rule

Suppose that y is a differentiable function of u and u is a differentiable function of x (i.e., $y = y(u)$ and $u = u(x)$), then y is a (differentiable) function of x by extension (i.e., $y = y(u(x))$) and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

→ Note: chain rule is used when we want to differentiate a function of another function.

Example(s):

(a) Differentiate with respect to x the function $y = (3x+4)^4$.

Solution
Let $y = u^4$, where $u = 3x+4$. Differentiating yields $\dfrac{dy}{du} = 4u^3$ and $\dfrac{du}{dx} = 3$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (4u^3)(3) = 12u^3 = 12(3x+4)^3$$

(b) Differentiate with respect to x the function $y = (x^2+3x)^7$.

Solution
Let $y = u^7$, where $u = x^2+3x$. Differentiating yields $\dfrac{dy}{du} = 7u^6$ and $\dfrac{du}{dx} = 2x+3$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (7u^6)(2x+3) = 7(x^2+3x)^6(2x+3)$$

(c) Find $\dfrac{dy}{dx}$ if $y = (1-3x^2)^5$.

Solution
Let $y = u^5$, where $u = 1-3x^2$. Differentiating yields $\dfrac{dy}{du} = 5u^4$ and $\dfrac{du}{dx} = -6x$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (5u^4)(-6x) = -30x(u^4) = -30x(1-3x^2)^4$$

(d) Find $\frac{dy}{dx}$ if $y = \left(\frac{1+2x}{1+x}\right)^2$.

Solution

Let $y = u^2$, where $u = \frac{1+2x}{1+x}$. Differentiating yields $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = \frac{(1+x)(2)-(1+2x)(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (2u)\left[\frac{1}{(1+x)^2}\right] = 2\left[\frac{1+2x}{1+x}\right]\left[\frac{1}{(1+x)^2}\right] = \frac{2(1+2x)}{(1+x)^3}$$

(e) Differentiate with respect to x the function $y = \sqrt{1+\sqrt{1+\sqrt{1+x}}}$.

Solution

Let $u = \sqrt{1+x}$, $v = \sqrt{1+u}$, and $w = \sqrt{1+v}$. Then, $y = w$. Differentiating yields $\frac{dy}{dw} = 1$, $\frac{du}{dx} = \frac{1}{2(1+x)^{\frac{1}{2}}}$, $\frac{dv}{du} = \frac{1}{2(1+u)^{\frac{1}{2}}}$, and $\frac{dw}{dv} = \frac{1}{2(1+v)^{\frac{1}{2}}}$. Therefore, chain rule yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dw}\cdot\frac{dw}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx} = (1)\left[\frac{1}{2(1+v)^{\frac{1}{2}}}\right]\left[\frac{1}{2(1+u)^{\frac{1}{2}}}\right]\left[\frac{1}{2(1+x)^{\frac{1}{2}}}\right] \\
&= \frac{1}{8(1+v)^{\frac{1}{2}}(1+u)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} = \frac{1}{8\left(\sqrt{1+v}\right)\left(\sqrt{1+u}\right)\left(\sqrt{1+x}\right)} \\
&= \frac{1}{8\left(\sqrt{1+\sqrt{1+\sqrt{1+x}}}\right)\left(\sqrt{1+\sqrt{1+x}}\right)\left(\sqrt{1+x}\right)}
\end{aligned}$$

→ Note: **(Direct chain rule)**

Consider the function $y = [f(x)]^n$. Then, direct chain rule yields

$$\boxed{\frac{dy}{dx} = n\,[f(x)]^{n-1}\cdot f'(x)}$$

For example, if $y = (1-3x^2)^5$ then DCR yields $\frac{dy}{dx} = 5(1-3x^2)^4(0-6x) = -30x(1-3x^2)^4$.

Exercise:

(a) Use chain rule to differentiate the following functions

i) $y = (3x^2+5)^3$

ii) $y = (3x^3+5x)^2$

iii) $y = (7x^2-4)^{\frac{1}{3}}$

iv) $y = (6x^2-4x)^{-2}$

v) $y = (3x^2-5)^{-\frac{2}{3}}$

vi) $y = (1+x^4-2x^3)^4(1-4x^2)^3$

(b) Find $\frac{dy}{dx}$ when $y = \sqrt{\frac{1+x}{1-x}}$. [ans: $\frac{dy}{dx} = \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$]

(c) Differentiate with respect to x the function $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$.

4.2 Derivative of trigonometric functions

☐ Derivative of $\sin x$ and $\cos x$

$$\frac{d}{dx}[\sin x] = \cos x \quad \text{and} \quad \frac{d}{dx}[\cos x] = -\sin x$$

Proof. i) SINE

Let $f(x) = \sin x$. Thus, from the first principle of differentiation and using the trigonometric identity $\sin(A+B) = \sin A \cos B + \sin B \cos A$, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \lim_{h\to 0} \frac{-\sin x(1-\cos h) + \sin h \cos x}{h} \\
&= -\sin x \left[\lim_{h\to 0} \frac{(1-\cos h)}{h}\right] + \cos x \left[\lim_{h\to 0} \frac{\sin h}{h}\right] = (-\sin x)(0) + (\cos x)(1) \\
&= \cos x
\end{aligned}$$

ii) COSINE

Similarly, let $f(x) = \cos x$. Thus, from the first principle of differentiation and using the trigonometric identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h\to 0} \frac{-\cos x(1-\cos h) - \sin x \sin h}{h} \\
&= -\cos x \left[\lim_{h\to 0} \frac{(1-\cos h)}{h}\right] - \sin x \left[\lim_{h\to 0} \frac{\sin h}{h}\right] = (-\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}$$

☐

Example(s):

(a) Differentiate the following functions wrt x: (i) $y = \sin(3x+2)$, (ii) $y = \cos^3 x$, (iii) $y = \sin(x^2)$, (iv) $y = x\sin(x)$, (v) $y = \dfrac{\sin x}{x}$, and (vi) $y = \cos^2(3x)$.

Solution

i) Given that $y = \sin(3x+2)$. Let $y = \sin(u)$, where $u = 3x+2$. Differentiating yields $\dfrac{dy}{du} = \cos u$ and $\dfrac{du}{dx} = 3$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(3) = 3\cos(3x+2)$$

ii) Given that $y = \cos^3 x$. Let $y = u^3$, where $u = \cos x$. Differentiating yields $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = -\sin x$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2)(-\sin x) = -3\sin x \cos^2 x$$

iii) Given that $y = \sin(x^2)$. Let $y = \sin(u)$, where $u = x^2$. Differentiating yields $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(2x) = 2x\cos(x^2)$$

iv) Given that $y = x\sin(x)$. Let $y = uv$, where $u = x$ and $v = \sin(x)$. Differentiating yields $u' = 1$ and $v' = \cos x$. Therefore, product rule yields

$$\frac{dy}{dx} = uv' + vu' = (x)(\cos x) + (\sin x)(1) = x\cos x + \sin x$$

v) Given that $y = \frac{\sin x}{x}$. Let $y = \frac{u}{v}$, where $u = \sin x$ and $v = x$. Differentiating yields $u' = \cos x$ and $v' = 1$. Therefore, quotient rule yields

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x)(\cos x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

vi) Given that $y = \cos^2(3x)$. Let $y = u^2$, where $u = \cos(3x)$. Differentiating yields $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = -3\sin(3x)$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u)[-3\sin(3x)] = -6\cos(3x)\sin(3x)$$

(b) If $y = \sqrt{1 + \sin x}$, show that $\frac{dy}{dx} = \frac{1}{2}\sqrt{1 - \sin x}$.

Solution

Let $y = u^{\frac{1}{2}}$, where $u = 1 + \sin x$. Differentiating yields $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx} = \cos x$. Therefore, chain rule yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(\cos x) = \frac{\cos x}{2\sqrt{u}} = \frac{\cos x}{2\sqrt{1+\sin x}} \\
&= \frac{1}{2}\frac{\cos x\sqrt{1-\sin x}}{(\sqrt{1+\sin x})(\sqrt{1-\sin x})} = \frac{1}{2}\frac{\cos x\sqrt{1-\sin x}}{\sqrt{1-\sin^2 x}} = \frac{1}{2}\frac{\cos x\sqrt{1-\sin x}}{\sqrt{\cos^2 x}} \\
&= \frac{1}{2}\sqrt{1-\sin x}
\end{aligned}$$

(c) Find $\frac{dy}{dx}$ if $y = \sin(\cos x)$.

Solution

Let $y = \sin u$, where $u = \cos x$. Differentiating yields $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = -\sin x$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(-\sin x) = -\cos(\cos x)\sin x$$

In general,

$$\frac{d}{dx}[\sin(f(x))] = f'(x)\cos(f(x)) \quad \text{and} \quad \frac{d}{dx}[\cos(f(x))] = -f'(x)\sin(f(x))$$

Exercise:

(a) Find $\frac{dy}{dx}$ if $y = \sin(\sqrt{x})$.

(b) If $y = \sqrt{\dfrac{1+\sin x}{1-\sin x}}$, show that $\dfrac{dy}{dx} = \dfrac{1}{1-\sin x}$.

(c) If m is a positive integer, find the differential coefficients with respect to x of (i) $\sin^m x$ and (ii) $\sin(x^m)$.

(d) Differentiate the following functions with respect to x: (i) $y = \sin 3x$, (ii) $y = \cos(x^2)$, (iii) $y = \sqrt{\sin 2x}$, (iv) $y = 4\sin^2(\frac{x}{2})$, (v) $y = \sin x \cos 2x$, (vi) $y = \dfrac{\cos 2x}{\sin 3x}$, and (vii) $y = 2\cos x + 2x \sin x - x^2 \cos x$.

☐ **Derivative of** $\tan x$, $\cot x$, $\sec x$ **and** $\operatorname{cosec} x$

I Tangent

Let $y = \tan x = \dfrac{\sin x}{\cos x}$. Differentiating using quotient rule yields

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Therefore, $\boxed{\dfrac{d}{dx}[\tan x] = \sec^2 x}$.

II Cotangent

Let $y = \cot x = \dfrac{\cos x}{\sin x}$. Differentiating using quotient rule yields

$$\frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\left[\sin^2 x + \cos^2 x\right]}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Therefore, $\boxed{\dfrac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x}$.

III Secant

Let $y = \sec x = \dfrac{1}{\cos x}$. Differentiating using quotient rule yields

$$\frac{dy}{dx} = \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

Therefore, $\boxed{\dfrac{d}{dx}[\sec x] = \sec x \tan x}$.

IV Cosecant

Let $y = \operatorname{cosec} x = \dfrac{1}{\sin x}$. Differentiating using quotient rule yields

$$\frac{dy}{dx} = \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

Therefore, $\boxed{\dfrac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x}$.

In summary, we have

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

Example(s):

(a) Differentiate the following functions with respect to x: (i) $y = \tan 2x$, (ii) $y = \cot 3x$, (iii) $y = 3\sec 2x$, (iv) $y = \sec x \tan x$, (v) $y = x^2 \cot x$, and (vi) $y = \dfrac{x}{\tan x}$.

Solution

i) Given that $y = \tan 2x$. Let $y = \tan u$, where $u = 2x$. Differentiating yields $\dfrac{dy}{du} = \sec^2 u$ and $\dfrac{du}{dx} = 2$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\sec^2 u)(2) = 2\sec^2(2x)$$

ii) Given that $y = \cot 3x$. Let $y = \cot u$, where $u = 3x$. Differentiating yields $\dfrac{dy}{du} = -\text{cosec}^2 u$ and $\dfrac{du}{dx} = 3$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-\text{cosec}^2 u)(3) = -3\text{cosec}^2(3x)$$

iii) Given that $y = 3\sec 2x$. Let $y = 3\sec u$, where $u = 2x$. Differentiating yields $\dfrac{dy}{du} = 3\sec u \tan u$ and $\dfrac{du}{dx} = 2$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3\sec u \tan u)(2) = 6\sec(2x)\tan(2x)$$

iv) Given that $y = \sec x \tan x$. Let $y = uv$, where $u = \sec x$ and $v = \tan x$. Differentiating yields $u' = \sec x \tan x$ and $v' = \sec^2 x$. Therefore, product rule yields

$$\frac{dy}{dx} = uv' + vu' = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$$

v) Given that $y = x^2 \cot x$. Let $y = uv$, where $u = x^2$ and $v = \cot x$. Differentiating yields $u' = 2x$ and $v' = -\text{cosec}^2 x$. Therefore, product rule yields

$$\frac{dy}{dx} = uv' + vu' = (x^2)(-\text{cosec}^2 x) + (\cot x)(2x) = 2x\cot x - x^2\text{cosec}^2 x$$

vi) Given that $y = \dfrac{x}{\tan x}$. Let $y = \dfrac{u}{v}$, where $u = x$ and $v = \tan x$. Differentiating yields $u' = 1$ and $v' = \sec^2 x$. Therefore, quotient rule yields

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(\tan x)(1) - (x)(\sec^2 x)}{\tan^2 x} = \frac{\tan x - x\sec^2 x}{\tan^2 x}$$

In general,

$$\frac{d}{dx}[\tan(f(x))] = f'(x)\sec^2(f(x)) \quad \text{and} \quad \frac{d}{dx}[\cot(f(x))] = -f'(x)\text{cosec}^2(f(x))$$

$$\frac{d}{dx}[\sec(f(x))] = f'(x)\sec(f(x))\tan(f(x)) \quad \text{and} \quad \frac{d}{dx}[\text{cosec}(f(x))] = -f'(x)\text{cosec}(f(x))\cot(f(x))$$

Exercise:

(a) Differentiate the following functions with respect to x: (i) $y = \sec^2 2x$, (ii) $y = 3\sec x \tan x$, (iii) $y = -\text{cosec}^2\left(\frac{1}{2}x\right)$, and (iv) $y = \dfrac{\sec x}{x}$.

(b) If $y = (\tan x + \sec x)^m$, where m is a positive integer. Show that $\dfrac{dy}{dx} = my\sec x$.

4.3 Derivative of exponential functions

An exponential function of x is defined by $y = e^x - -(*)$ or $y = \exp(x)$. Consider an exponential function of the form $(*)$. Then,

$$\frac{d}{dx}[e^x] = e^x$$

Proof. Given $y(x) = e^x \quad \Rightarrow \quad y(x+h) = e^{x+h} = e^x e^h$. Thus, by the first principle of differentiation, we have

$$y'(x) = \lim_{h\to 0}\left[\frac{y(x+h) - y(x)}{h}\right] = \lim_{h\to 0}\left[\frac{e^x e^h - e^x}{h}\right] = e^x \lim_{h\to 0}\left[\frac{e^h - 1}{h}\right]$$

From the table, we have

h	0.0001	0.001	0.01	0.1	-0.01	-0.001
$\frac{e^h - 1}{h}$	1.0005			1.05		

Hence, $\lim_{h\to 0}\left[\frac{e^h - 1}{h}\right] = 1$

Therefore, $y'(x) = e^x(1) = e^x$. □

In general, suppose $y = e^{f(x)}$. Let $y = e^u$, where $u = f(x)$. Differentiating yields $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$. By chain rule, we have $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot f'(x) = f'(x)e^{f(x)}$. Therefore,

$$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$$

Example(s):

(a) Find $\frac{dy}{dx}$ given (i) $y = e^{-6x}$ and (ii) $y = e^{x^2}$

Solution

i) Given $y = e^{-6x}$. Let $u = -6x \quad \Rightarrow \quad y = e^u$. Differentiating we get $\frac{du}{dx} = -6$ and $\frac{dy}{du} = e^u$.
Hence, chain rule yields $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -6e^u = -6e^{-6x}$. Therefore, $\frac{dy}{dx} = -6e^{-6x}$.

ii) Given $y = e^{x^2}$. Let $u = x^2 \quad \Rightarrow \quad y = e^u$. Differentiating we get $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = e^u$.
Hence, chain rule yields $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2xe^u = 2xe^{x^2}$. Therefore, $\frac{dy}{dx} = 2xe^{x^2}$.

(b) Differentiate the following functions with respect to x: (i) $y = 2e^{-3x} + e^{4x}$ and (ii) $y = 2e^{\sin 3\theta}$.

Solution

Using direct chain rule, we have

i) $\frac{dy}{dx} = 2\frac{d}{dx}\left[e^{-3x}\right] + \frac{d}{dx}\left[e^{4x}\right] = 2\left[e^{-3x}\frac{d}{dx}(-3x)\right] + \left[e^{4x}\frac{d}{dx}(4x)\right] = -6e^{-3x} + 4e^{4x}$.

ii) $\frac{dy}{d\theta} = 2\frac{d}{d\theta}\left[e^{\sin 3\theta}\right] = 2\left[e^{\sin 3\theta}\frac{d}{d\theta}(\sin 3\theta)\right] = 2e^{\sin 3\theta}(3\cos 3\theta) = 6e^{\sin 3\theta}\cos 3\theta$.

(c) If $y = e^{-2x}\cos 4x$, find $\frac{dy}{dx}$.

Solution

Product rule yields

$$\frac{dy}{dx} = e^{-2x}\frac{d}{dx}[\cos 4x] + \cos 4x\frac{d}{dx}\left[e^{-2x}\right] = -4e^{-2x}\sin 4x - 2e^{-2x}\cos 4x$$

Exercise:

1. Find $\frac{dy}{dx}$ given:

 i) $y = \sqrt{x}e^{\sqrt{x}} + e^{\sqrt{x^2+1}}$

 ii) $y = \frac{e^{2x}}{1 + x^2e^{3x}}$

 iii) $y = (6 + e^{3x}\cos 4x)^4$

 iv) $y = e^{\sin 5x} + 2x^2e^{\cos 3x}$

 v) $y = \left(e^{3x^2+6x}\right)\cos\left(e^x + e^{-x}\right)$

 vi) $y = e^{\tan(4+\sin 3x)} + \frac{1}{e^{2x} + e^{5x}} + 6.$

4.4 Derivative of natural logarithmic functions

A natural logarithm function of x is logarithm of x to base e. For example, $\ln|x| = \log_e x$. Thus, x can be rewritten as $x = e^{\ln x}$. Similarly, $y = e^{\ln y}$, $a = e^{\ln a}$, etc. Suppose $y = \ln x$. Then,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Proof. Given $y = \ln x$. Taking exponential on both sides yields $e^y = x$. Differentiating both sides with respect to x yields $e^y\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$. Therefore, $\frac{dy}{dx} = \frac{1}{x}$. □

In general, suppose $y = \ln[f(x)]$. Then, $e^y = f(x)$. Differentiating both sides with respect to x yields

$$e^y\frac{dy}{dx} = f'(x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{f'(x)}{e^y} = \frac{f'(x)}{f(x)}$$

Therefore,

$$\frac{d}{dx}\{\ln[f(x)]\} = \frac{f'(x)}{f(x)}$$

Example(s):

(a) Find $\frac{dy}{dx}$ given: (i) $y = \ln(x^2)$, (ii) $y = \ln(\cos 2x)$, (iii) $y = \ln\left(\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right)$.

 Solution

 i) Given $y = \ln(x^2)$. Let $u = x^2 \quad \Rightarrow \quad y = \ln(u)$. Differentiating yields $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = \frac{1}{u} = \frac{1}{x^2}$. Hence, chain rule yields $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$. Therefore,

$$\frac{dy}{dx} = \frac{2}{x}$$

 ii) Let $y = \ln(u)$ where $u = \cos 2x$. Differentiating yields $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = -2\sin 2x$. Hence, chain rule yields $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{-2\sin 2x}{u} = \frac{-2\sin 2x}{\cos 2x} = -2\tan 2x$. Therefore,

$$\frac{dy}{dx} = -2\tan 2x$$

iii) Given $y = \ln\left(\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right) \Rightarrow y = \ln\left[\frac{(x^2+1)^{1/2}}{(x^3+1)^{1/3}}\right] = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}\ln(x^3+1)$.

Thus, $y = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}\ln(x^3+1)$. Differentiating both sides with respect to x yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2+1\right)\right] - \frac{1}{3}\frac{d}{dx}\left[\ln\left(x^3+1\right)\right] \\
&= \frac{1}{2(x^2+1)}\frac{d}{dx}\left(x^2+1\right) - \frac{1}{3(x^3+1)}\frac{d}{dx}\left(x^3+1\right) = \frac{1}{2(x^2+1)}(2x) - \frac{1}{3(x^3+1)}\left(3x^2\right) \\
&= \frac{x}{(x^2+1)} - \frac{x^2}{(x^3+1)} = \frac{x(x^3+1) - x^2(x^2+1)}{(x^2+1)(x^3+1)} = \frac{x^4+x-x^4-x^2)}{(x^2+1)(x^3+1)} \\
&= \frac{x-x^2}{(x^2+1)(x^3+1)}
\end{aligned}$$

(b) Find $\frac{dy}{dx}$ given that $y = \sin(\ln 2x)$.

Solution

Let $y = \sin u$, where $u = \ln 2x$. Thus, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\cos u)\left[\frac{1}{2x}\cdot\frac{d}{dx}(2x)\right] = (\cos u)\left[\frac{1}{x}\right] = \frac{\cos u}{x} = \frac{\cos(\ln 2x)}{x}$$

(c) Find $\frac{dy}{dx}$ given $y = \sqrt{\frac{1+x}{1-x}}$.

Solution

Given $y = \sqrt{\frac{1+x}{1-x}}$. Taking natural logarithm on both sides yields

$$\ln y = \ln\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$$

Thus, $\ln y = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$. Differentiating with respect to x yields

$$\begin{aligned}
\frac{d}{dx}[\ln y] &= \frac{1}{2}\left\{\frac{d}{dx}[\ln(1+x)] - \frac{d}{dx}[\ln(1-x)]\right\} \\
\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\left\{\frac{1}{(1+x)}\frac{d}{dx}(1+x) - \frac{1}{(1-x)}\frac{d}{dx}(1-x)\right\} = \frac{1}{2}\left\{\frac{1}{1+x} + \frac{1}{1-x}\right\} \\
&= \frac{1}{1-x^2}
\end{aligned}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{1-x^2}y = \frac{1}{1-x^2}\sqrt{\frac{1+x}{1-x}} = \frac{1}{(1+x)(1-x)}\cdot\frac{(1+x)^{1/2}}{(1-x)^{1/2}} = \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$$

Exercise:

(a) i) If $y = (\sqrt{x-1})e^x \ln x$, find $\frac{dy}{dx}$. [ans: $\frac{dy}{dx} = \frac{e^x}{2x\sqrt{x-1}}\{(2x^2-x)\ln x + 2x - 2\}$]

ii) Find the gradient of the curve $y = \ln(\sqrt{1+\sin 2x})$ at the point where $x = \frac{\pi}{2}$. [ans: = -1]

(b) Find $\frac{dy}{dx}$ given:

i) $y = \ln\sqrt{2x+6}$

ii) $y = \sqrt{x}\ln(\sqrt{x})$

iii) $y = \dfrac{1 + 2x^2 \ln 3x}{1 + \sqrt{\sec(\ln 2x)}}$

iv) $y = \dfrac{1}{1 + 2\ln 46x} - \dfrac{1}{\sin(\ln(15x^2))}$

v) $y = \left(2x^2 + \ln\sqrt{x}\right)^6 (1 + 2x\sec 2x)^3$

vi) $y = \cot\left(\ln 2x + e^{3x}\right)$

LECTURE 6

4.5 Implicit differentiation

An implicit function is a function where the dependent variable y is not expressed explicitly in terms of the independent variable x (i.e., a function where y is not the subject of the formula). To find $\dfrac{dy}{dx}$, follow these steps:

i) Differentiate x normally

ii) Apply direct chain rule in differentiating y

iii) Collect like terms and make $\dfrac{dy}{dx}$ the subject

Example(s):

1. Find $\dfrac{dy}{dx}$ given $x^4 + y^5 = 125$.

   Solution

   Differentiating the given equation implicitly with respect to x, we get

$$4x^3 + 5y^4\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{4x^3}{5y^4}$$

2. Find $\dfrac{dy}{dx}$ given $y + xy + y^2 = 2$.

   Solution

   Differentiating the given equation implicitly with respect to x, we get

$$\frac{dy}{dx} + x\frac{dy}{dx} + (1)y + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y}{1 + x + 2y}$$

3. Find $\dfrac{dy}{dx}$ when $y^3 - 3x^2y + 2x^3 = 0$.

   Solution

   Differentiating the given equation implicitly with respect to x, we get

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} - 6xy + 6x^2 = 0 \quad \Rightarrow \quad (3y^2 - 3x^2)\frac{dy}{dx} = 6xy - 6x^2$$

   Therefore,

$$\frac{dy}{dx} = \frac{6x(y - x)}{3(y^2 - x^2)} = \frac{6x(y - x)}{3(y + x)(y - x)} = \frac{2x}{y + x}$$

4. If $y^2 - 2y\sqrt{1+x^2} + x^2 = 0$, show that $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1+x^2}}$.

   Solution

   Differentiating the given equation implicitly with respect to x, we get

   $$2y\frac{dy}{dx} - 2\sqrt{1+x^2}\frac{dy}{dx} - 2y\left[\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\right](2x) + 2x = 0 \quad \Rightarrow \quad \left(2y - 2\sqrt{1+x^2}\right)\frac{dy}{dx} = \frac{2xy}{\sqrt{1+x^2}} - 2x$$

   Therefore,

   $$\frac{dy}{dx} = \frac{\dfrac{2xy}{\sqrt{1+x^2}} - 2x}{2y - 2\sqrt{1+x^2}} = \frac{2x\left(y - \sqrt{1+x^2}\right)}{2\sqrt{1+x^2}\left(y - \sqrt{1+x^2}\right)} = \frac{x}{\sqrt{1+x^2}}$$

Exercise:

1. Find $\dfrac{dy}{dx}$ given:

   (a) $xy^3 - 2x^2y^2 + x^4 = 1$
   (b) $x^2 \sin y - y\cos x = 10x^3$
   (c) $x\cos y - y^2 \sin x = 2$
   (d) $e^{xy^2} = 10\left(x^2 + y^2\right)$
   (e) $\ln(x^2 + \sqrt{y}) = \sin(xy^2)$
   (f) $\tan y \sin x = \cos(xy)$
   (g) $\cos(x+y)\sin(x-y) = 20x^2$
   (h) $y^2 e^{x^2 y} = \dfrac{30}{\sqrt{xy}}$
   (i) $\ln\left(\dfrac{x+y}{x^2 y}\right) = 10x^2$

4.6 Differentiation of other forms of exponential functions

Consider exponential functions of the form

I: **A constant raised to a function** (e.g., $y = 10^{\tan 3x}$)
In this case, introduce natural logarithm on both sides first. On the left differentiate implicitly and on the right hand side differentiate normally.

Example(s):

(a) Find $\dfrac{dy}{dx}$ given: (i) $y = a^{x^2}$ where a is a constant, (ii) $y = 3^{-x^2+6x+10}$, and (iii) $y = 4^{\sin 5x}$.

Solution

i) Given $y = a^{x^2}$. Taking natural logarithm on both sides yields

$$\ln y = x^2 \ln a$$

Differentiating with respect to x yields

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x^2 \ln a] \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2x\ln a \quad \Rightarrow \quad \frac{dy}{dx} = 2xy\ln a$$

Therefore, $\dfrac{dy}{dx} = 2xa^{x^2}\ln a$.

ii) Given $y = 3^{-x^2+6x+10}$. Taking natural logarithm on both sides yields

$$\ln y = (-x^2 + 6x + 10) \ln 3$$

Differentiating with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = (-2x + 6) \ln 3 \quad \Rightarrow \quad \frac{dy}{dx} = (-2x + 6) y \ln 3$$

Therefore, $\dfrac{dy}{dx} = (-2x + 6)3^{-x^2+6x+10} \ln 3$.

iii) Given $y = 4^{\sin 5x}$. Taking natural logarithm on both sides yields

$$\ln y = \sin 5x \ln 4$$

Differentiating with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = 5 \cos 5x \ln 4 \quad \Rightarrow \quad \frac{dy}{dx} = (5 \cos 5x) y \ln 4$$

Therefore, $\dfrac{dy}{dx} = (5 \cos 5x)4^{\sin 5x} \ln 4$.

Exercise:

i) Find $\dfrac{dy}{dx}$ given $y = a^x + b^x$, where a and b are constants. [ans: $\dfrac{dy}{dx} = a^x \ln a + b^x \ln b$]

II: **A function raised to a function** (e.g., $y = x^{\tan 3x}$)
In this case, introduce natural logarithm on both sides first. On the left differentiate implicitly and on the right hand side differentiate using product rule.

Example(s):

(a) Find $\dfrac{dy}{dx}$ given: (i) $y = x^x$, (ii) $y = (\tan x)^x$, (iii) $y = (\sin 4x)^{x^2}$, and (iv) $y = \left(e^{x^2}\right)^x$.

Solution

i) Given $y = x^x$. Taking natural logarithm on both sides yields

$$\ln y = x \ln x$$

Differentiating with respect to x yields

$$\begin{aligned} \frac{d}{dx}[\ln y] &= \frac{d}{dx}[x \ln x] \\ \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}(x) = x \cdot \frac{1}{x} + \ln x \\ &= 1 + \ln x \end{aligned}$$

Therefore, $\dfrac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$.

ii) Given $y = (\tan x)^x$. Taking natural logarithm on both sides yields

$$\ln y = x \ln(\tan x)$$

Differentiating with respect to x yields

$$\begin{aligned} \frac{d}{dx}[\ln y] &= \frac{d}{dx}[x \ln(\tan x)] \\ \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}[\ln(\tan x)] + \ln(\tan x)\frac{d}{dx}(x) = x \cdot \frac{\sec^2 x}{\tan x} + \ln(\tan x) \\ &= \frac{x \sec^2 x}{\tan x} + \ln(\tan x) \end{aligned}$$

Therefore, $\dfrac{dy}{dx} = y\left(\dfrac{x \sec^2 x}{\tan x} + \ln(\tan x)\right) = (\tan x)^x \left(\dfrac{x \sec^2 x}{\tan x} + \ln(\tan x)\right)$.

iii) Given $y = (\sin 4x)^{x^2}$. Taking natural logarithm on both sides yields

$$\ln y = x^2 \ln(\sin 4x)$$

Differentiating with respect to x yields

$$\begin{aligned} \frac{d}{dx}[\ln y] &= \frac{d}{dx}[x^2 \ln(\sin 4x)] \\ \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= x^2 \frac{d}{dx}[\ln(\sin 4x)] + \ln(\sin 4x)\frac{d}{dx}(x^2) = x^2 \cdot \frac{4\cos 4x}{\sin 4x} + 2x\ln(\sin 4x) \\ &= 4x^2 \cot 4x + 2x \ln(\sin 4x) \end{aligned}$$

Therefore, $\frac{dy}{dx} = y\left[4x^2 \cot 4x + 2x\ln(\sin 4x)\right] = (\sin 4x)^{x^2}\left[4x^2 \cot 4x + 2x \ln(\sin 4x)\right]$.

Exercise:

(b) Differentiate the following functions with respect to x.

i) $y = x^{\cos x}$. [ans: $\frac{dy}{dx} = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$]

ii) $y = (\sin x)^x$. [ans: $\frac{dy}{dx} = (\sin x)^x\,(\ln(\sin x) + x\cot x)$]

iii) $y = (x + x^2)^x$. [ans: $\frac{dy}{dx} = (x+x^2)^x\left(\ln(x+x^2) + \frac{1+2x}{1+x}\right)$]

iv) $y^x = x$. [ans: $\frac{dy}{dx} = \frac{y}{x}\left(\frac{1}{x} - \ln y\right)$]

v) $x^y = \sin x$. [ans: $\frac{dy}{dx} = \frac{1}{\ln x}\left(\cot x - \frac{y}{x}\right)$]

vi) $y^{\sin x} = \sqrt{x}$. [ans: $\frac{dy}{dx} = \frac{y}{\sin x}\left(\frac{1}{2x} - \cot x \ln y\right)$]

(b) Find $\frac{dy}{dx}$ given:

i) $y = (\cos 3x)^{\sin 3x}$

ii) $y = a^x + \sin(2^x)$

iii) $y = (\cos x)^x + 2x^2$

iv) $y = \sin(a^{2x} + 3^x)$

III: **Derivatives of other logarithmic functions** e.g., $y = \log_2(3x^2 + 1)$
To find $\frac{dy}{dx}$, first convert the logarithm to index notation then introduce natural logarithm on both sides.

Example(s):

(a) Find $\frac{dy}{dx}$ given: (i) $y = \log_2(3x^2+1)$, (ii) $y = \log_{\sin x} x$, and (iii) $y = \log_x(\cos 3x)$

Solution

i) Given $y = \log_2(3x^2+1)$. In index notation, we have $2^y = 3x^2 + 1$. Taking natural logarithm on both sides yields

$$y \ln 2 = \ln(3x^2+1)$$

Differentiating with respect to x yields

$$\frac{d}{dx}[y\ln 2] = \frac{d}{dx}[\ln(3x^2+1)] \quad \Rightarrow \quad \frac{dy}{dx}\ln 2 = \frac{6x}{3x^2+1}$$

Therefore, $\frac{dy}{dx} = \frac{6x}{(3x^2+1)\ln 2}$.

ii) Given $y = \log_{\sin x} x$. In index notation, we have $(\sin x)^y = x$. Taking natural logarithm on both sides yields

$$y \ln(\sin x) = \ln x$$

Differentiating with respect to x yields

$$y \frac{\cos x}{\sin x} + \ln(\sin x)\frac{dy}{dx} = \frac{1}{x} \quad \Rightarrow \quad y \cot x + \ln(\sin x)\frac{dy}{dx} = \frac{1}{x}$$

Therefore, $\dfrac{dy}{dx} = \left(\dfrac{1}{x} - y \cot x\right) \dfrac{1}{\ln(\sin x)}$.

iii) Given $y = \log_x(\cos 3x)$. In index notation, we have $x^y = \cos 3x$. Taking natural logarithm on both sides yields

$$y \ln x = \ln(\cos 3x)$$

Differentiating with respect to x yields

$$\frac{y}{x} + \ln(x)\frac{dy}{dx} = \frac{-3\sin 3x}{\cos 3x} \quad \Rightarrow \quad \frac{y}{x} + \ln(x)\frac{dy}{dx} = -3\tan 3x$$

Therefore, $\dfrac{dy}{dx} = \left(-\dfrac{y}{x} - 3\tan 3x\right) \dfrac{1}{\ln x}$.

LECTURE 7

4.7 Inverse trigonometric functions

$$y = \sin^{-1} x = \text{arcsin } x, \quad y = \cos^{-1} x = \text{arccos } x, \quad y = \tan^{-1} x = \text{arctan } x$$

$$y = \text{cosec}^{-1} x = \text{arccosec } x, \quad y = \sec^{-1} x = \text{arcsec } x, \quad y = \cot^{-1} x = \text{arccot } x.$$

→ Note: $\sin^{-1} x \neq \dfrac{1}{\sin x}$, $\cos^{-1} x \neq \dfrac{1}{\cos x}$, etc.

4.7.1 Derivative of inverse trigonometric functions

To find $\dfrac{dy}{dx}$, follow these steps

i) Introduce the trigonometric function corresponding to the given inverse on both sides of the given equation

ii) Differentiate implicitly on the left hand side and differentiate normally on the right hand side.

iii) Make $\dfrac{dy}{dx}$ the subject.

To find a suitable form of the trigonometric function in the denominator,

- ☐ put the given equation in place of y, or
- ☐ replace the denominator by either making use of trigonometric identities or draw a right angled triangle and find the missing side using Pythagoras theorem, as follows:

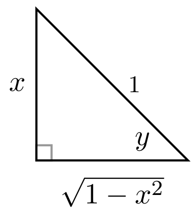
For example,

I Let $y = \sin^{-1} x \quad \Rightarrow \quad \sin y = x$. Differentiating both sides with respect to x yields

$$\cos y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

☐ Formula 1: putting $y = \sin^{-1} x$ in place of y yields $\dfrac{dy}{dx} = \dfrac{1}{\cos\left(\sin^{-1} x\right)}$

- ☐ Formula 2:
  - ☐ using the identity $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \sqrt{1-\sin^2 y}$. Thus, $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$. Putting $\sin y = x$ yields $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, or
  - ☐ using the right angled triangle



From $\sin y = x$, we have opposite is x and hypotenuse is 1. From Pythagoras theorem, the adjacent is given by $\sqrt{1-x^2}$. From the diagram, $\cos y = \sqrt{1-x^2}$. Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Therefore, $\boxed{\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}}$.

II Let $y = \cos^{-1} x \quad \Rightarrow \quad \cos y = x$. Differentiating both sides with respect to x yields

$$-\sin y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

Therefore, $\boxed{\frac{d}{dx}\left[\cos^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}}}$.

III Let $y = \tan^{-1} x \quad \Rightarrow \quad \tan y = x$. Differentiating both sides with respect to x yields

$$\sec^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Therefore, $\boxed{\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}}$.

IV Let $y = \cot^{-1} x \quad \Rightarrow \quad \cot y = x$. Differentiating both sides with respect to x yields

$$-\text{cosec}^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

Therefore, $\boxed{\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}}$.

V Let $y = \sec^{-1} x \quad \Rightarrow \quad \sec y = x$. Differentiating both sides with respect to x yields

$$\sec y \tan y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \left(\sqrt{\sec^2 y - 1}\right)} = \frac{1}{x\sqrt{x^2-1}}$$

Therefore, $\boxed{\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{x\sqrt{x^2-1}}}$.

VI Let $y = \text{cosec}^{-1} x \quad \Rightarrow \quad \text{cosec } y = x$. Differentiating both sides with respect to x yields

$$-\text{cosec } y \cot y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\text{cosec } y \cot y} = \frac{-1}{\text{cosec } y \left(\sqrt{\text{cosec}^2 y - 1}\right)} = \frac{-1}{x\sqrt{x^2-1}}$$

Therefore, $\boxed{\frac{d}{dx}\left[\text{cosec}^{-1} x\right] = \frac{-1}{x\sqrt{x^2-1}}}$.

Example(s):

(a) Find $\dfrac{dy}{dx}$ if $y = \sin^{-1}(2x^2 + x + 1)$.

Solution

Let $y = \sin^{-1} u$, where $u = 2x^2 + x + 1$. Differentiating yields $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx} = 4x + 1$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)(4x+1) = \frac{4x+1}{\sqrt{1-(2x^2+x+1)^2}}$$

(b) Find $\dfrac{dy}{dx}$ if $y = \cos^{-1}(2x + 1)$.

Solution

Let $y = \cos^{-1} u$, where $u = 2x+1$. Differentiating yields $\dfrac{dy}{du} = \dfrac{-1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx} = 2$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{-1}{\sqrt{1-u^2}}\right)(2) = \frac{-2}{\sqrt{1-(2x+1)^2}}$$

(c) Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}(\cos x + x)$.

Solution

Let $y = \tan^{-1} u$, where $u = \cos x + x$. Differentiating yields $\dfrac{dy}{du} = \dfrac{1}{1+u^2}$ and $\dfrac{du}{dx} = -\sin x + 1$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1}{1+u^2}\right)(-\sin x + 1) = \frac{-\sin x + 1}{1 + (\cos x + x)^2}$$

(d) Differentiate $y = x \sin^{-1} x$ with respect to x.

Solution

Let $y = uv$, where $u = x$ and $v = \sin^{-1} x$. Differentiating yields $u' = 1$ and $v' = \dfrac{1}{\sqrt{1-x^2}}$. Therefore, product rule yields

$$\frac{dy}{dx} = uv' + vu' = (x)\left(\frac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1} x)(1) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$$

(e) Find $\dfrac{d\theta}{dt}$ when (i) $\theta = \cos^{-1}(1 - 2t^2)$ and (ii) $\theta = \sin^{-1}(2t^3 - 1)$.

Solution

i) Given that $\theta = \cos^{-1}(1 - 2t^2)$. Let $\theta = \cos^{-1} u$, where $u = 1 - 2t^2$. Differentiating yields $\dfrac{d\theta}{du} = \dfrac{-1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dt} = -4t$. Therefore, chain rule yields

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{du} \cdot \frac{du}{dt} = \left(\frac{-1}{\sqrt{1-u^2}}\right)(-4t) = \frac{4t}{\sqrt{1-(1-2t^2)^2}} = \frac{4t}{\sqrt{1-1+4t^2-4t^4}} \\ &= \frac{4t}{\sqrt{4t^2(1-t^2)}} = \frac{4t}{2t\sqrt{1-t^2}} = \frac{2}{\sqrt{1-t^2}}\end{aligned}$$

ii) Given that $\theta = \sin^{-1}(2t^3 - 1)$. Let $\theta = \sin^{-1} u$, where $u = 2t^3 - 1$. Differentiating yields $\dfrac{d\theta}{du} = \dfrac{1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dt} = 6t^2$. Therefore, chain rule yields

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{du}\cdot\frac{du}{dt} = \left(\frac{1}{\sqrt{1-u^2}}\right)(6t^2) = \frac{6t^2}{\sqrt{1-(2t^3-1)^2}} = \frac{6t^2}{\sqrt{1-4t^6+4t^3-1}} \\ &= \frac{4t}{\sqrt{4t^2(t-t^4)}} = \frac{6t^2}{2t\sqrt{t-t^4}} = \frac{3t}{\sqrt{t-t^4}}\end{aligned}$$

(f) If $y = \left(\tan^{-1} x\right)^2$, prove that $\dfrac{d}{dx}\left\{(1+x^2)\dfrac{dy}{dx}\right\} = \dfrac{2}{1+x^2}$.

Proof. Let $y = u^2$, where $u = \tan^{-1} x$. Differentiating yields $\dfrac{dy}{du} = 2u$ and $\dfrac{du}{dx} = \dfrac{1}{1+x^2}$. Therefore, chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (2u)\left(\frac{1}{1+x^2}\right) = \frac{2\tan^{-1}x}{1+x^2}$$

Now,

$$\frac{d}{dx}\left\{(1+x^2)\frac{dy}{dx}\right\} = \frac{d}{dx}\left\{(1+x^2)\frac{2\tan^{-1}x}{1+x^2}\right\} = \frac{d}{dx}\left\{2\tan^{-1}x\right\} = \frac{2}{1+x^2}$$

□

Exercise:

(a) i) If $y = \sin^{-1}(\cos x)$, show that $\dfrac{dy}{dx} = -1$.

ii) If $y = \sin^{-1}(3x - 4x^3)$, show that $\sqrt{1-x^2}\dfrac{dy}{dx} = 3$.

iii) If $u = \theta^2 + \left(\sin^{-1}\theta\right)^2 - 2\theta\sqrt{1-\theta^2}\sin^{-1}\theta$, show that $\sqrt{1-\theta^2}\dfrac{du}{d\theta} = 4\theta^2\sin^{-1}\theta$.

(b) Find the derivative of the following functions: (i) $y = x^2\left(\sin^{-1})x\right)^2$, (ii) $y = \sqrt{\tan^{-1}x}$, and (iii) $y = \sin\left(\tan^{-1}x\right)$.

(c) Find $\dfrac{dy}{dx}$ given that $y = \sin^{-1}\sqrt{x}$. [ans: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x-x^2}}$]

(d) Find $\dfrac{dy}{dx}$ given:

i) $y = e^{\sin^{-1}(3x)} + \dfrac{1}{2x + \cos^{-1}(4x)}$

ii) $y = 2^x + \cos^{-1}(4x)$

iii) $y = \ln\left(x^2 + 2^x\right)$

iv) $y = \dfrac{\sin^{-1}(3x)}{2^{3x} + \sin 3x}$

v) $y = \operatorname{cosec}^{-1}(3x)$

vi) $y = x^2\operatorname{cosec}^{-1}(4x)$

vii) $y = x^x\sin^{-1}(2x)$

CAT 1

Lecture 8

4.8 Parametric differentiation

If both x and y are defined as functions of another variable (parameter), say t, i.e., $x = x(t), \;\; y = y(t)$, then

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \div \left(\frac{dx}{dt}\right) = \frac{dy/dt}{dx/dt}}$$

Example(s):

1. Find $\dfrac{dy}{dx}$, in terms of the parameter t, when (a) $x = at^2, \;\; y = 2at$, (b) $x = (t+1)^2, \;\; y = (t^2 - 1)$, and (c) $x = \cos^{-1}(3t), \;\; y = \sin^{-1}(3t)$.

   Solution

   (a) $\dfrac{dx}{dt} = 2at, \;\; \dfrac{dy}{dt} = 2a$. Therefore, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2a}{2at} = \dfrac{1}{t}$.

   (b) $\dfrac{dx}{dt} = 2(t+1), \;\; \dfrac{dy}{dt} = 2t$. Therefore, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{2(t+1)} = \dfrac{t}{t+1}$.

   (c) Rewrite as $\cos x = 3t, \;\; \sin y = 3t$. Differentiating with respect to t yields $-\sin x \dfrac{dx}{dt} = 3$ and $\cos y \dfrac{dy}{dt} = 3$. Hence, $\dfrac{dx}{dt} = -\dfrac{3}{\sin x} = -\dfrac{3}{\sqrt{1-9t^2}}$ and $\dfrac{dy}{dt} = \dfrac{3}{\cos y} = \dfrac{3}{\sqrt{1-9t^2}}$. Therefore, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \left(\dfrac{3}{\sqrt{1-9t^2}}\right) \div \left(-\dfrac{3}{\sqrt{1-9t^2}}\right) = -1$.

Exercise:

(a) Find $\dfrac{dy}{dx}$ given:

   i) $x = \ln(2t^2), \;\; y = \ln(4 + t^2)$

   ii) $x = 2^t, \;\; y = 2^{-t}$

   iii) $x = \tan^{-1}(2t), \;\; y = \sec^{-1}(2t)$

   iv) $x = t\sin(t^2), \;\; y = t^3\cos(t^2)$

   v) $x = \dfrac{t^2}{1+t^2}, \;\; y = \dfrac{1-t^2}{1+t^2}$

   vi) $x = e^t \cos 2t, \;\; y = e^{-t} \sin 2t$

   vii) $x = \theta - \sin 2\theta, \;\; y = \theta + \cos 2\theta$

   viii) $x = a\cos^3\theta, \;\; y = b\sin^3\theta$

4.9 Higher order derivatives

Suppose $y(x)$ is an $n-$times differentiable function of x. Then,

First derivative of $y(x)$ is given by	$\dfrac{dy}{dx} = y'(x)$
Second derivative of $y(x)$ is given by	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = y''(x)$
Third derivative of $y(x)$ is given by	$\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3} = y'''(x)$
$\vdots$	$\vdots$
nth derivative of $y(x)$ is given by	$\dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right) = \dfrac{d^ny}{dx^n} = y^{(n)}(x)$

Example(s):

1. Given that $y = 4x^3 - 6x^2 - 9x + 1$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, and $\frac{d^4y}{dx^4}$.

Solution

$$\frac{dy}{dx} = 12x^2 - 12x - 9, \qquad \frac{d^2y}{dx^2} = 24x - 12, \qquad \frac{d^3y}{dx^3} = 24, \qquad \frac{d^4y}{dx^4} = 0.$$

2. Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ when (a) $y = x^{10}$ and (b) $y = \cos 2x$.

Solution

(a) $\frac{dy}{dx} = 10x^9, \qquad \frac{d^2y}{dx^2} = 90x^8, \qquad \frac{d^3y}{dx^3} = 720x^7.$

(b) $\frac{dy}{dx} = -2\sin 2x, \qquad \frac{d^2y}{dx^2} = -4\cos 2x, \qquad \frac{d^3y}{dx^3} = 8\sin 2x.$

3. If $y = \frac{\cos x}{x}$, prove that $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$.

Proof. Quotient rule yields $\frac{dy}{dx} = \frac{x\frac{d}{dx}[\cos x] - \cos x\frac{d}{dx}[x]}{x^2} = \frac{-x\sin x - \cos x}{x^2}$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{x^2\frac{d}{dx}[-x\sin x - \cos x] - (-x\sin x - \cos x)\frac{d}{dx}[x^2]}{x^4} \\
&= \frac{x^2(-x\cos x - \sin x + \sin x) + (x\sin x + \cos x)(2x)}{x^4} = \frac{-x^3\cos x + 2x^2\sin x + 2x\cos x}{x^4} \\
&= \frac{-x^2\cos x + 2x\sin x + 2\cos x}{x^3}
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y &= \frac{-x^2\cos x + 2x\sin x + 2\cos x}{x^3} + \frac{2}{x}\left(\frac{-x\sin x - \cos x}{x^2}\right) + \frac{\cos x}{x} \\
&= \frac{-x^2\cos x + 2x\sin x + 2\cos x - 2x\sin x - 2\cos x + x^2\cos x}{x^3} \\
&= 0
\end{aligned}$$

□

4. Given that u and v are functions of x, show that $\frac{d^3}{dx^3}(uv) = \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}$.

Proof.

$$\begin{aligned}
\frac{d^3}{dx^3}(uv) &= \frac{d}{dx}\left[\frac{d^2}{dx^2}(uv)\right] = \frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{d}{dx}(uv)\right\}\right] = \frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{du}{dx}v + u\frac{dv}{dx}\right\}\right] \quad \text{(product rule)} \\
&= \frac{d}{dx}\left[\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right] = \frac{d^3u}{dx^3}v + \frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3} \\
&= \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}
\end{aligned}$$

□

Exercise:

1. Find $\frac{d^2y}{dx^2}$ when (a) $y = (1+4x+x^2)\sin x$, (b) $y = x\tan^{-1}x$, (c) $y = \frac{x^2}{1+x}$, (d) $y = (3x - \sin 2x)^2$, and (e) $y = \ln(3x^3 + 4x - 1) + xe^{x^2}$.

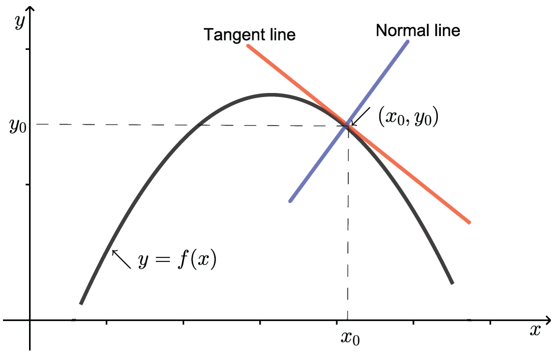
2. If $y = \dfrac{\sin x}{x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Hence, prove that $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + (x^2+2)y = 0$.

Lecture 9

5 Applications of differentiation

5.1 Equation of a tangent line and normal line to a curve

Consider the diagram below



A tangent line to a curve is a line that touches the curve at one point, say (x_0, y_0), while a normal line to a curve is a line perpendicular to the tangent line and passes through the point (x_0, y_0).

1. The rate of change of y with respect to x, i.e., $\dfrac{dy}{dx} = y'(x)$, gives the gradient function to the curve $y = f(x)$.

2. If $y'(x)$ (or $\dfrac{dy}{dx}$) is evaluated at point $x = x_0$, the result is the gradient of the tangent line at the point $x = x_0$.

3. Since the normal line is perpendicular to the tangent line, the product of their gradients must be equal to -1, i.e.,
$$\boxed{m_1 \times m_2 = -1},$$
where m_1 and m_2 represents the gradient of the tangent line and normal line, respectively.

4. To find the equation of a straight line, we require a known point (x_0, y_0), a general point (x, y), which must all lie on the line, and the gradient, $m = \dfrac{\Delta y}{\Delta x}$, of the line. Thus, the equation of a straight line is given by
$$\boxed{\frac{y - y_0}{x - x_0} = m}$$

Example(s):

1. Find the equation of the tangent line and normal line to the curve $x^2 + 2xy + 3y^2 = 17$ at point $(1, 2)$.

   Solution

   Clearly, the point $(1, 2)$ lies on the given curve. Now, differentiating the given curve implicitly with respect to x yields $2x + 2x\dfrac{dy}{dx} + 2y + 6y\dfrac{dy}{dx} = 0$. Therefore,
$$\frac{dy}{dx} = \frac{-(x+y)}{x+3y}$$

The gradient of the tangent line at point $(1, 2)$ is $\dfrac{dy}{dx} = \dfrac{-(1+2)}{1+3(2)} = -\dfrac{3}{7} = m_1$. Thus, the equation of the tangent line at point $(1, 2)$ is

$$\frac{y-2}{x-1} = -\frac{3}{7} \quad \Rightarrow \quad y = -\frac{3}{7}x + \frac{17}{7}$$

The gradient of the normal line at $(1, 2)$ is $m_2 = -\dfrac{1}{m_1} = \dfrac{7}{3}$. Thus, the equation of the normal line at $(1, 2)$ is

$$\frac{y-2}{x-1} = \frac{7}{3} \quad \Rightarrow \quad y = \frac{7}{3}x - \frac{1}{3}$$

2. Find the equation of tangent line and normal line to the following curves at the indicated points.

   (a) $2e^{-x} + e^y = 3e^{x-y}$ at point (0,0).

   Solution

   Clearly, the point $(0, 0)$ lies on the given curve. Now, differentiating the given curve implicitly with respect to x yields $-2e^{-x} + e^y y' = 3(1 - y')e^{x-y}$. Therefore,

   $$y' = \frac{2e^{-x} + 3e^{x-y}}{e^y + 3e^{x-y}}$$

   The gradient of the tangent line at point $(0, 0)$ is $\dfrac{dy}{dx} = \dfrac{2e^{-0} + 3e^0}{e^0 + 3e^0} = \dfrac{2+3}{1+3} = \dfrac{5}{4} = m_1$. Thus, the equation of the tangent line at point $(0, 0)$ is

   $$\frac{y-0}{x-0} = \frac{5}{4} \quad \Rightarrow \quad y = \frac{5}{4}x$$

   The gradient of the normal line at $(0, 0)$ is $m_2 = -\dfrac{1}{m_1} = -\dfrac{4}{5}$. Thus, the equation of the normal line at $(0, 0)$ is

   $$\frac{y-0}{x-0} = -\frac{4}{5} \quad \Rightarrow \quad y = -\frac{4}{5}x$$

   (b) $xy = 6e^{2x-3y}$ at point (3,2). [hint: $m_1 = 10/21$]

   (c) $x = \dfrac{t^2}{1+t}$, $y = \dfrac{t^3}{1-t}$ at $t = 2$. [hint: $m_1 = 7/2$]

   (d) $y = a\cos^3 t$, $x = a\sin^3 t$ at $t = \dfrac{\pi}{4}$. [hint: $m_1 = -1$]

3. The parametric equations of a curve are $x = 3(2\theta - \sin 2\theta)$, $y = 3(1 - \cos 2\theta)$. The tangent and normal to the curve at the point P where $\theta = \dfrac{\pi}{4}$ meet the $y-$axis at L and M, respectively. Show that the area of triangle PLM is $\dfrac{9}{4}(\pi - 2)^2$.

   Solution

   $\dfrac{dx}{d\theta} = 3(2 - 2\cos 2\theta)$ and $\dfrac{dy}{d\theta} = 6\sin 2\theta$. Therefore, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{6\sin 2\theta}{3(2 - 2\cos 2\theta)} = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$.

   When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{\sin\left(\frac{\pi}{2}\right)}{1 - \cos\left(\frac{\pi}{2}\right)} = \dfrac{1}{1-0} = 1$. Hence,

   ☐ gradient of the tangent at P is 1. Now, the value of x when $\theta = \dfrac{\pi}{4}$ is $x = 3\left[\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right] = \dfrac{3\pi - 6}{2}$. The value of y when $\theta = \dfrac{\pi}{4}$ is $y = 3\left[1 - \cos\left(\frac{\pi}{2}\right)\right] = 3$. Thus, the co-ordinate of point P is $\left(\dfrac{3\pi - 6}{2}, 3\right)$.

☐ Equation of the tangent at P is

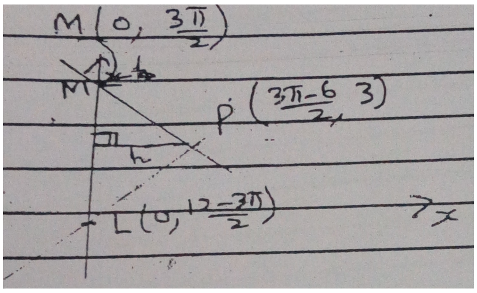
$$\frac{y-3}{x-\dfrac{3\pi-6}{2}}=1 \quad\Rightarrow\quad y=x-\frac{3\pi-6}{2}+3 \quad\Rightarrow\quad y=x+\frac{12-3\pi}{2}$$

Thus, the $y-$intercept is $y=\dfrac{12-3\pi}{2}$. Hence, the co-ordinate of point L is $\left(0,\dfrac{12-3\pi}{2}\right)$.

☐ gradient of the normal at P is -1. Thus, equation of the normal at P is

$$\frac{y-3}{x-\dfrac{3\pi-6}{2}}=-1 \quad\Rightarrow\quad y=-x+\frac{3\pi-6}{2}+3 \quad\Rightarrow\quad y=-x+\frac{3\pi}{2}$$

Thus, the $y-$intercept is $y=\dfrac{3\pi}{2}$. Hence, the co-ordinate of point M is $\left(0,\dfrac{3\pi}{2}\right)$.



Height of triangle PLM is $h=\dfrac{3\pi-6}{2}$ and the base length is $LM=\dfrac{3\pi}{2}-\dfrac{12-3\pi}{2}=3\pi-6$. Thus, area of triangle PLM

$$=\frac{1}{2}(3\pi-6)\left(\frac{3\pi-6}{2}\right)=\frac{9}{4}(\pi-2)^2$$

Exercise:

1. Find the equation of tangent and normal to the following curves
   (a) $12(x^2+y^2)=25xy$ at point (3,4)
   (b) $x^2y=x+2$ at point (2,1)
   (c) $xy=\ln\left(\frac{x}{y}\right)$ at point (1,3)
   (d) $xy=6e^{2x-3y}$ at point (3,2)
   (e) $\dfrac{1}{x+1}+\dfrac{1}{y+1}=1$ at point (1,1)
   (f) $x=\dfrac{t^2+1}{t+3t^2},\ y=\dfrac{t^3-7}{t+3t^2}$ at $t=1$
2. Show that the equation of the tangent to $x^2+xy+y=0$ at the point (x_1,y_1) is $(2x_1+y_1)x+(x_1+1)y+y_1=0$.
3. Show that the equation of the tangent at (x_1,y_1) to the curve $ax^2+by^2+cxy+dx=0$ is $ax_1x+by_1y+\frac{1}{2}c(y_1x+x_1y)+\frac{1}{2}d(x_1+x)=0$.

Lecture 10

5.2 Linear approximation/small changes

Linear approximation is a technique used to estimate values of some functions close to some known results. The equation of tangent line at the point (x_0,y_0) can be used to approximate the function $y(x)$ close to this point. Now, the gradient of tangent line to the curve $y(x)$ at point (x_0,y_0) is denoted by

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=y'(x_0)$$

Thus, the equation of the tangent line at this point is $\dfrac{\Delta y}{\Delta x} = \text{gradient} \quad \Rightarrow \quad \dfrac{y - y_0}{x - x_0} = y'(x_0)$. Therefore, for x close to x_0 and denoting $y_0 = y(x_0)$, we have the following approximation of $y(x)$:

$$\boxed{y(x) \approx y(x_0) + y'(x_0)(x - x_0).}$$

→ Notes:

(1) This formula comes from the slope $\dfrac{dy}{dx} \approx \dfrac{y - y_0}{x - x_0}$ for x close to x_0.

(2) This is also equivalent to taking $\dfrac{\Delta y}{\Delta x} \approx \dfrac{\mathrm{d}y}{\mathrm{d}x}$, i.e. the differential as an approximation of the increment.

Example(s):

(a) Use linear approximation to estimate (i) $\sqrt{26}$ and (ii) $\sqrt[4]{80}$.

Solution

i) Note that $\sqrt{26} = 26^{\frac{1}{2}}$. Let $y(x) = x^{\frac{1}{2}} \quad \Rightarrow \quad y'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. Take $x_0 = 25$ (a value close to 26 and has exact square root), we have $y(x_0) = (25)^{\frac{1}{2}} = 5$ and $y'(x_0) = \frac{1}{2}(25)^{-\frac{1}{2}} = \frac{1}{10}$. By linear approximation, we have

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0) = 5 + \frac{1}{10}(x - 25)$$

Plugging in $x = 26$ yields

$$y(26) \approx 5 + \frac{(26 - 25)}{10} = 5 + \frac{1}{10} = 5.1$$

Therefore, $\sqrt{26} \approx 5.1$.

ii) Note that $\sqrt[4]{80} = 80^{\frac{1}{4}}$. Let $y(x) = x^{\frac{1}{4}} \quad \Rightarrow \quad y'(x) = \frac{1}{4}x^{-\frac{3}{4}}$. Take $x_0 = 81$ (a value close to 80 and has exact fourth root), we have $y(x_0) = (81)^{\frac{1}{4}} = 3$ and $y'(x_0) = \frac{1}{4}(81)^{-\frac{3}{4}} = \frac{1}{108}$. By linear approximation, we have

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0) = 3 + \frac{1}{108}(x - 81)$$

Plugging in $x = 80$ yields

$$y(80) \approx 3 + \frac{(80 - 81)}{108} = 3 - \frac{1}{108} = \frac{324 - 1}{108} = \frac{323}{108} \approx 2.9907$$

Therefore, $\sqrt[4]{80} \approx 2.9907$.

(b) Use differentials to approximate (i) $\cos(44^o)$ and (ii) $\sin(60^o 1')$.

Solution

i) Let $y = \cos x \quad \Rightarrow \quad \dfrac{dy}{dx} = -\sin x$. Take $x_0 = 45^o = \dfrac{\pi}{4}$ (a value close to 44^o and its cosine can be obtained without using SMP table or a calculator), we have $y(x_0) = \cos(45^o) = \dfrac{1}{\sqrt{2}}$ and $y'(x_0) = -\sin(45^o) = -\dfrac{1}{\sqrt{2}}$. By linear approximation, we have

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

Plugging in $x = 44^o \equiv \frac{11\pi}{45}$ radians yields

$$y\left(\frac{11\pi}{45}\right) \approx \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{11\pi}{45} - \frac{\pi}{4}\right) = 0.7194$$

Therefore, $\sin(44^o) \approx 0.7194$.

ii) Let $y = \sin x \quad \Rightarrow \quad \frac{dy}{dx} = \cos x$. Take $x_0 = 60^o = \frac{\pi}{3}$ (a value close to $60^o 1'$ and its sine can be obtained without using SMP table or a calculator), we have $y(x_0) = \sin(60^o) = \frac{\sqrt{3}}{2}$ and $y'(x_0) = \cos(60^o) = \frac{1}{2}$. By linear approximation, we have

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

Plugging in $x = 60^o 1' \equiv \left(\frac{\pi}{3} + 0.0003\right)$ radians yields

$$y\left(\frac{\pi}{3} + 0.0003\right) \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi}{3} + 0.0003 - \frac{\pi}{3}\right) = 0.86618$$

Therefore, $\sin(60^o 1') \approx 0.86618$.

Exercise:

(a) Estimate $\sqrt[5]{30}$ by linear approximation. [ans: $\sqrt[5]{30} \approx 1.975$]

(b) Find the approximate value of $80^{\frac{3}{4}}$ using linear approximation. [ans: $80^{\frac{3}{4}} \approx 26.75$]

(c) Find the cube root of 24 without using a calculator.

(d) Find the linearization of the function $y = \sqrt{x+3}$ at $x_0 = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

(e) Use linear approximation to estimate $\ln(1.1)$. You must make an appropriate choice of where to center your approximation. Draw a picture illustrating your approximation and write and explanation of why you chose to base your approximation where you did (In other words, explain your choice of x_0).

(f) Use a linear approximation to estimate $y(4.1)$, given that $y(4) = 2$ and $\frac{dy}{dx} = \sqrt{x^2 + 20}$.

5.3 Related rates

If a variable x is a function of time t, the time rate of change of x is given by $\frac{dx}{dt}$. When two or more variables, all functions of t, are related by an equation, the relation between their rates of change may be obtained by differentiating the equation with respect to t. For example, if $z^2 = x^2 + y^2$ then differentiating implicitly with respect to t yields $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. To solve related rates problem, do the following:

i) Assign symbols to all quantities given and their respective rates of change. Use a sketch where necessary.

ii) Write an equation relating all the variables whose rates of change are given or are to be determined.

iii) Apply chain rule of differentiation to differentiate implicitly both sides of the equation with respect to time t

iv) Substitute into the resulting equation all the known variables and their rates of change. Then, solve for the required rate of change.

Example(s):

1. Sand is falling in a conical pipe at a rate of 100 m^3 per minute. Find the rate of change of the height when the height is 10m. (Assume that the coarseness of the sand is such that the height is equal to the radius).

   Solution

   ☐ Step I: Let V be the volume of the conical pile, h the height and r the radius. Given $\frac{dV}{dt} = 100\ \text{m}^3/\text{min}$. We are required to find $\frac{dh}{dt}$ when $h = 10\text{m}$ and $r = h$.

   ☐ Step II: At time t the coned has volume $V = \frac{1}{3}\pi r^2 h$. Putting $r = h$ yields $V = \frac{1}{3}\pi h^3$.

   ☐ Step III: Differentiating implicitly with respect to t, we obtain

   $$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{\pi h^2}\frac{dV}{dt}$$

   ☐ Step IV: Substituting $\frac{dV}{dt} = 100$ and $h = 10$ yields $\frac{dh}{dt} = \frac{1}{\pi(10)^2}(100) = \frac{1}{\pi} = 0.318\ \text{m/min}$. Therefore, the height is increasing at the rate of 0.318 meters per minute.

2. Gas is escaping from a spherical balloon at the rate of 900 cm^3/s. How fast is the surface area shrinking when the radius is 360 cm.

   Solution

   ☐ Step I: Let V be the volume of the sphere, S the surface area and r the radius. Given $\frac{dV}{dt} = -900\ \text{m}^3/\text{min}$. We are required to find $\frac{dS}{dt}$ when $r = 360\text{m}$.

   ☐ Step II: At time t the coned has volume $V = \frac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$.

   ☐ Step III: Differentiating implicitly with respect to t, we obtain

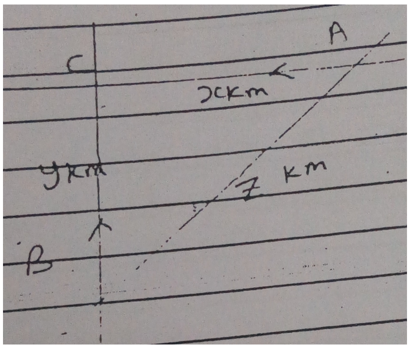
   $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

   Thus, $\frac{dS}{dt} = 8\pi r\left(\frac{1}{4\pi r^2}\frac{dV}{dt}\right) \quad \Rightarrow \quad \frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt}$.

   ☐ Step IV: Substituting $\frac{dV}{dt} = -900$ and $r = 360$ yields $\frac{dS}{dt} = \frac{2}{360}(-900) = -5$. Therefore, the surface area is decreasing at the rate of 5cm^2 per second.

3. Car A is traveling west at 50km/h and car B is traveling north at 60km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3km and car B is 0.4km from the intersection?

   Solution



☐ Step I: Let x and y be the distance of car A and B from C at any time t, respectively. Let z be the distance between car A and B at any time t. Given $\frac{dx}{dt} = 50\text{km/h}$ and $\frac{dy}{dt} = 60\text{km/h}$. We are required to find $\frac{dz}{dt}$ when $x = 0.3\text{km}$ and $y = 0.4\text{km}$.

☐ Step II: At time t the distance between A and B is $z^2 = x^2 + y^2$.

☐ Step III: Differentiating implicitly with respect to t, we obtain

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \qquad (*)$$

☐ Step IV: When $x = 0.3\text{km}$ and $y = 0.4\text{km}$, we have

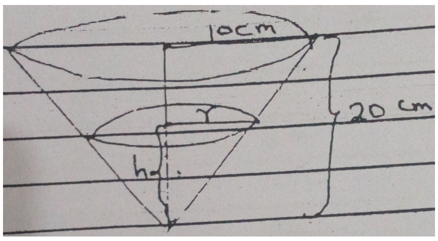
$$z = \sqrt{x^2 + y^2} = \sqrt{(0.3)^2 + (0.4)^2} = 0.5\text{km}$$

Substituting $x = 0.3\text{km}$, $y = 0.4\text{km}$, $z = 0.5\text{km}$, $\frac{dx}{dt} = -50\text{km/h}$ and $\frac{dy}{dt} = -60\text{km/h}$ in equation $(*)$ yields $\frac{dz}{dt} = \frac{1}{0.5}((0.3)(-50) + (0.4)(-60)) = -78$. Therefore, the two cars are approaching each other at a rate of 78km/h.

Exercise:

1. Water is running out at the rate of $5\text{cm}^3/\text{s}$. If the radius of the base of the funnel is 10cm and the altitude is 20cm, find the rate at which the water level is dropping when it is 5cm from the top.

Solution



Let r be the radius, h the height of the surface of the water, and V the volume of water in the cone at time t. The volume of the cone at time t is given by the equation

$$V = \frac{1}{3}\pi r^2 h$$

But by similar triangles, we have $\frac{r}{10} = \frac{h}{20} \Rightarrow r = \frac{h}{2}$. Therefore, $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12}h^3$. Differentiating implicitly with respect to t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $\frac{dV}{dt} = -5\text{cm}^3/\text{s}$ and $h = 20 - 5 = 15\text{cm}$, we get $\frac{dh}{dt} = \frac{4}{\pi(15)^2}(-5) = -\frac{4}{45\pi}$. Therefore, the water level is dropping at the rate of $\frac{4}{45\pi}$ cm/s.

2. Gas is escaping from a spherical balloon at the rate of $0.02m^3/s$. How fast is the surface area shrinking when the radius is $4m$.

Solution

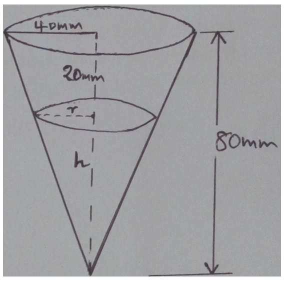
Given $\frac{dV}{dt} = -0.02m^3/s$. At time t, the sphere has radius r. So, volume, $V = \frac{4}{3}\pi r^3$ and surface area, $S = 4\pi r^2$. Now,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ and } \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \quad \Rightarrow \quad \frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt}$$

When $r = 4m$, $\frac{dS}{dt} = \frac{2}{4m}(-0.02m^3/s) = -0.01m^2/s$.

3. Water is running out of a conical funnel at the rate of $1000m^3/s$. If the radius of the base of funnel is $40mm$ and the altitude is $80mm$, find the rate at which the water level is dropping when it is $20mm$ from the top.

Solution



Let r be the radius and h be the height of the surface of the water at time t and V be the volume of the water in the cone. So, $V = \frac{1}{3}\pi r^2 h$. Given further that $r = 40mm$ when $h = 80mm$ $\Rightarrow$ $\frac{r}{h} = \frac{40}{80} \quad \Rightarrow \quad r = \frac{h}{2}$. Now,

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi rh\frac{dr}{dt} \text{ and } \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{1}{3}\pi(h/2)^2\frac{dh}{dt} + \frac{2}{3}\pi(h/2)h\frac{1}{2}\frac{dh}{dt} = \frac{1}{12}\pi h^2\frac{dh}{dt} + \frac{2}{12}\pi h^2\frac{dh}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$

Therefore,

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$

When $\frac{dV}{dt} = -1000m^3/s$, $h = 80mm - 20mm = 60mm = 0.06m$. Hence,

$$-1000m^3/s = \frac{1}{4}\pi(0.06)^2m^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{-1000m^3/s \times 4}{\pi(0.06)^2m^2} = \frac{-10}{9\pi}m/s$$

4. Sands falling from a chute form a conical pile whose altitude is equal to $\frac{4}{3}$ the radius of the base.

   (a) How fast is the volume increasing when the radius of the base is 0.3m and is increasing at the rate of $0.025m/s$.

   Solution

   Let r be the radius of the base, h be the height, and V be the volume of the conical pile at time t. So, $V = \frac{1}{3}\pi r^2 h$. Given further that $h = \frac{4}{3}r$. Now,

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi rh\frac{dr}{dt} \text{ and } \frac{dh}{dt} = \frac{4}{3}\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{1}{3}\pi r^2 \left(\frac{4}{3}\frac{dr}{dt}\right) + \frac{2}{3}\pi r \left(\frac{4}{3}r\right)\frac{dr}{dt} = \frac{4}{9}\pi r^2 \frac{dr}{dt} + \frac{8}{9}\pi r^2 \frac{dr}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt}$$

Therefore,

$$\frac{dV}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt}$$

When $r = 0.3m$, $\frac{dr}{dt} = 0.025m/s$. Hence,

$$\frac{dV}{dt} = \frac{4}{3}\pi(0.3m)^2(0.025m/s) = 0.003\pi m^3/s$$

(b) How fast is the radius increasing when it is 0.6m and the volume is increasing at the rate of $0.024m^3/s$.

Solution

When $r = 0.6m$, $\frac{dV}{dt} = 0.024m^3/s$. Hence,

$$0.024m^3/s = \frac{4}{3}\pi(0.6m)^2\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{0.024m^3/s \times 3}{4\pi \times (0.6m)^2} = \frac{1}{20\pi}m/s$$

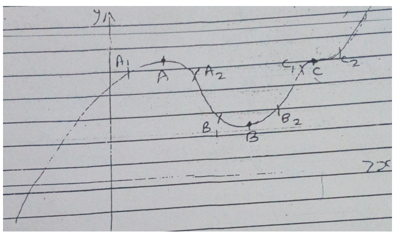
5. A spherical balloon is blown up so that its volume increases at a constant rate of $2\text{cm}^3/\text{s}$. Find the rate of increase of the radius when the volume of the balloon is 50cm^3.

6. Ink is dropped onto blotting paper forming a circular stain which increases in area at the rate of $5\text{cm}^2/\text{s}$. Find the rate of change of the radius when the area is 30cm^2.

7. A rectangle is twice as long as it is broad. Find the rate of change of the perimeter when the breadth of the rectangle is 1m and its area is changing at the rate of $18\text{cm}^2/\text{s}$, assuming the expansion is uniform.

5.4 Turning points/Stationary points/Critical points/Extrema

The points at which the slope (gradient) of a curve is zero are called stationary/turning points. For example, consider the following curve $y = f(x)$:



A- a maximum turning point

B- a minimum turning point

C- a point of inflection

To classify the stationary values, consider the points A_1 and A_2, B_1 and B_2, C_1 and C_2 which are left and right of A, B, and C, respectively, and close to them.

(1) **First derivative test** (or sign test)

This test relies on the sign of $\frac{dy}{dx}$ just to the LHS and just to the RHS of the turning point. Consider the behaviour of the gradient $\frac{dy}{dx}$ at points A, B and C.

☐ For A (a maximum point)

at point A_1, $\frac{dy}{dx}$ is positive (+ve)

at point A, $\frac{dy}{dx}$ is zero (0)

at point A_2, $\frac{dy}{dx}$ is negative (-ve)

☐ For B (a minimum point)

at point B_1, $\dfrac{dy}{dx}$ is negative (-ve)

at point B, $\dfrac{dy}{dx}$ is zero (0)

at point B_2, $\dfrac{dy}{dx}$ is positive (+ve)

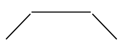
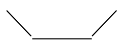
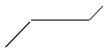
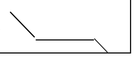
☐ For C (a point of inflection)

at point C_1, $\dfrac{dy}{dx}$ is positive (+ve)

at point C, $\dfrac{dy}{dx}$ is zero (0)

at point C_2, $\dfrac{dy}{dx}$ is positive (+ve)

Summary:

	Maximum	**Minimum**	**Inflection**
Sign of $\dfrac{dy}{dx}$ when moving through a stationary value	$+\ 0\ -$ 	$-\ 0\ +$ 	$+\ 0\ +$ OR $-\ 0\ -$  OR 

(2) **Second derivative test**: first compute $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Now, when passing through point A, $\dfrac{dy}{dx}$ changes from +ve to -ve i.e., $\dfrac{dy}{dx}$ decreases as x increases

$$\frac{d^2y}{dx^2} \text{ is negative (i.e., } \frac{d^2y}{dx^2} < 0)$$

Similarly, when passing through point B, $\dfrac{dy}{dx}$ changes from -ve to +ve i.e., $\dfrac{dy}{dx}$ increases as x increases

$$\frac{d^2y}{dx^2} \text{ is positive (i.e., } \frac{d^2y}{dx^2} > 0)$$

Summary:

	Maximum	**Minimum**	**Inflection**
Sign of $\dfrac{d^2y}{dx^2}$ at a turning point	negative i.e., $\left(\dfrac{d^2y}{dx^2} < 0\right)$	positive i.e., $\left(\dfrac{d^2y}{dx^2} > 0\right)$	zero i.e., $\left(\dfrac{d^2y}{dx^2} = 0\right)$

In **summary**, the steps for finding critical points are?

i) Find the first derivative

ii) Set it to zero

iii) Find the turning points

iv) Use the second derivatives to check whether the points you found are maxima/minima/points of inflection.

Example(s):

1. Find the stationary points of the following curves and classify them.

(a) $y = x^4 + 4x^3 - 6$.

Solution
Differentiating the given curve, we obtain

$$\frac{dy}{dx} = 4x^3 + 12x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 12x^2 + 24x$$

At a stationary point, $\frac{dy}{dx} = 0$, i.e., $4x^3 + 12x^2 = 0$. Solving yields $x = 0$ or $x = -3$. The value of y at $x = 0$ is $y = (0)^4 + 4(0)^3 - 6 = -6$. Similarly, the value of y at $x = -3$ is $y = (-3)^4 + 4(-3)^3 - 6 = -33$. So the turning points are $(0, -6)$ and $(-3, -33)$. Using the second derivative test, we classify the points as follows:

- ☐ When $x = -3$, we have $\frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3) = 36 > 0$. Therefore, the point $(-3, -33)$ is a minimum point.
- ☐ When $x = 0$, we have $\frac{d^2y}{dx^2} = 12(0)^2 + 24(0) = 0$. Therefore, the point $(0, -6)$ is a point of inflection.

(b) $y = x^2(x + 1)$.

Solution
Differentiating the given curve, we obtain

$$\frac{dy}{dx} = 3x^2 + 2x \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x + 2$$

At a stationary point, $3x^2 + 2x = 0$. Solving yields $x = 0$ or $x = -\frac{2}{3}$. The value of y at $x = 0$ is $y = (0)^2(0 + 1) = 0$. Also, The value of y at $x = -\frac{2}{3}$ is $y = \left(-\frac{2}{3}\right)^2\left(-\frac{2}{3} + 1\right) = \frac{4}{27}$. So the turning points are $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{27}\right)$. Using the second derivative test, we classify the points as follows:

- ☐ When $x = 0$, we have $\frac{d^2y}{dx^2} = 6(0) + 2 = 2 > 0$. Therefore, $(0, 0)$ is a minimum point.
- ☐ When $x = -\frac{2}{3}$, we have $\frac{d^2y}{dx^2} = 6\left(-\frac{2}{3}\right) + 2 = -2 < 0$. Therefore, $\left(-\frac{2}{3}, \frac{4}{27}\right)$ is a maximum point.

Exercise:

1. Find the maximum and minimum values of the function $y = 2\sin t + \cos 2t$. [ans: max point $(\frac{\pi}{6}, \frac{3}{2})$, min point $(\frac{\pi}{2}, 1)$]
2. Find the turning points and point of inflection on the curve $y = x^5 - 5x^4 + 5x^3 - 1$. [ans: max point $(1, 0)$, min point $(3, -28)$, point of inflection (0,-1)]
3. Discuss the nature of the points on the curve $y = 3x^4 - 8x^3 - 24x^2 + 96x$ at which the tangent to the curve is parallel to the $x-$axis.
4. Find the nature of the stationary points of the function $y = 3x^5 + 6x^4 - 4x^3 + 1$.
5. Show that the minimum value of the curve $y = a\sec\theta - b\tan\theta$ is $\sqrt{a^2 - b^2}$.

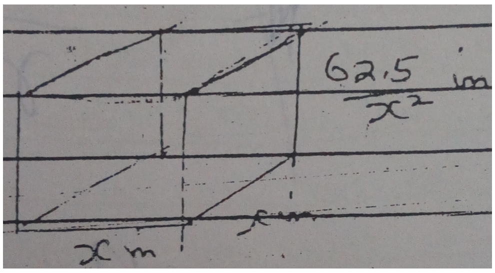
☐ **Areas where concept of maxima and minima is applied**

5.4.1 Optimization

Example(s):

1. A box with a square base and an open top is to have volume 62.5 in^3. Neglect the thickness of the material used to make the box, and find the dimensions that will minimize the amount of material used.

Solution



Let the base width be x in. Thus, the height is $\frac{62.5}{x^2}$ in. The surface area is given by

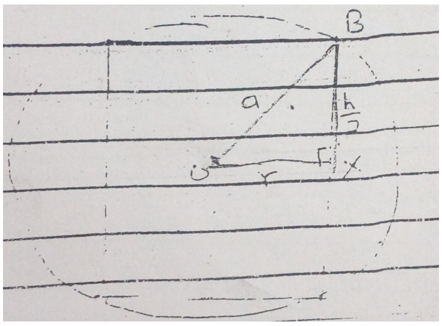
$$A = x^2 + 4x\left(\frac{62.5}{x^2}\right) = x^2 + \frac{250}{x}$$

Thus, $\frac{dA}{dx} = 2x - \frac{250}{x^2}$ and $\frac{d^2A}{dx^2} = 2 + \frac{500}{x^3}$.

For maximum or minimum area, $\frac{dA}{dx} = 0$. That is, $2x - \frac{250}{x^2} = 0$. Solving yields $x = 5$. At $x = 5$, $\frac{d^2A}{dx^2} = 2 + \frac{500}{(5)^3} = 6 > 0$. Therefore, the box has a minimum area when its base is 5 in by 5 in and height is $\frac{62.5}{(5)^2} = 2.5$ in.

2. Find the height of the right circular cylinder of greatest volume which can be cut from a sphere of radius a.

Solution



Let the radius of the cylinder be r and its height be h. From triangle OBX, Pythagoras theorem yields

$$r^2 + \frac{h^2}{4} = a^2$$

Thus, $r^2 = a^2 - \frac{h^2}{4}$. The volume of the cylinder is given by

$$V = \pi r^2 h = \pi\left(a^2 - \frac{h^2}{4}\right)h = \pi\left(a^2 h - \frac{h^3}{4}\right)$$

Thus, $\frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right)$ and $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$. For maximum or minimum volume, $\frac{dV}{dh} = 0$. That is, $\pi\left(a^2 - \frac{3}{4}h^2\right) = 0$. Solving yields $h = \frac{2a}{\sqrt{3}}$. At $h = \frac{2a}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -\frac{3\pi a}{\sqrt{3}} < 0$. Therefore, the cylinder has a maximum volume when its height is $\frac{2a}{\sqrt{3}}$.

3. Find the values of x and y that will maximize the function $f(x, y) = xy$ subject to the constraint $4x + 2y = 40$.

Solution

Given $4x + 2y = 40$, we have $y = 20 - 2x$. Substituting in $f(x, y) = xy$, we get

$$f(x) = x(20 - 2x) = 20x - 2x^2$$

Differentiating with respect to x, we have $f'(x) = 20 - 4x$. At maxima or minima, $f'(x) = 0$, i.e., $20 - 4x = 0 \quad \Rightarrow \quad x = 5$. Substituting in the given constraint, we get $y = 10$. To classify the optimal value, we use second derivative test. Now, $f''(x) = -4 < 0$. Therefore, $x = 5$ and $y = 10$ will maximize xy.

Exercise:

1. Find the height of the right circular cone of maximum volume, given that the sum of the height and radius of the base is 0.12m [ans: $h = 0.04$m]
2. Find the dimensions of the rectangle of greatest area which can be inscribed in a circle of radius r. [ans: a square of side $\sqrt{2}r$]
3. A manufacturer wants to design an open box having square base and surface area of 108 square meters. Find the dimensions of the box that will give maximum volume. [hint: $x^2 + 4xh = 108, V = 27x - \frac{x^3}{4}$, ans: length x= 6m and height h= 3m]
4. ABCD is a square ploughed field of side 132m, with a path along its perimeter. A man can walk at 8 km/h along the path, but only at 5 km/h across the field. He starts from A along AB, leaves AB at a point P, and walks straight from P to C. Find the distance of P from A, if the time taken is the least possible.
5. (a) Choose x and y to maximise xy subject to the constraint $3x + y = 60$. (Note: you do not need to confirm that your solution is a maximum).

   (b) Choose x and y to maximise xy^2 subject to the constraint $x + y = 200$. [ans: $x = 200/3$ and $y = 400/3$]

5.4.2 Economics: cost, revenue and profit

- ☐ The cost function, $C(x)$, is the (total) cost incurred in producing x units of a commodity.
- ☐ Marginal cost (MC) is the rate of change of the cost function with respect to the number of units produced, i.e., MC $= \frac{dC}{dx}$. It represents the extra cost incurred in producing one extra unit when the level of production is already at x.
- ☐ Average cost (AC) is given by AC $= \frac{C(x)}{x}$.
- ☐ Revenue, $R(x)$ is the (total) revenue received when x units of a given commodity are produced and sold at a unit price $p(x)$ (or demand function). Thus, $R(x) = x \cdot p(x)$, where x is the number of units produced and sold.
- ☐ Marginal revenue, MR $= \frac{dR}{dx}$.
- ☐ Profit function, $P(x)$ or $\Pi(x)$, is given by total revenue minus total cost, i.e., $P(x) = R(x) - C(x)$.
- ☐ Marginal profit, MP $= \frac{dP}{dx}$.
- ☐ Average profit $= \frac{P(x)}{x}$.

At maxima or minima, we have MC $= \frac{dC}{dx} = 0$, or MR $= \frac{dR}{dx} = 0$ or MP $= \frac{dP}{dx} = 0$. The values of x are the critical points.

→ Note: **Break Even point** refers to the point in which total cost and total revenue are equal, i.e., $C(x) = R(x)$.

Example(s):

1. In marketing a certain commodity, a business has discovered that the demand for the commodity is represented by $p(x) = \dfrac{50}{\sqrt{x}}$. The cost of producing x units of the commodity is given by $C(x) = 0.5x + 500$. Find the price per unit that will yield maximum profit. (Note: p is in dollars)

   Solution

   The profit function is given by

$$P(x) = xp(x) - C(x) = x\frac{50}{\sqrt{x}} - (0.5x + 500) = 50\sqrt{x} - 0.5x - 500$$

   The marginal profit (MP) is given by MP $= \dfrac{dP}{dx} = \dfrac{25}{\sqrt{x}} - 0.5$. At maxima or mimima, $\dfrac{dP}{dx} = 0$. That is,

$$\frac{25}{\sqrt{x}} - 0.5 = 0 \quad \Rightarrow \quad x = 2500$$

   We need to test if this value of x will lead to maximum profit. Now, $\dfrac{d^2P}{dx^2} = -12.5x^{-3/2}$. When $x = 2500$, $P''(x) = -12.5(2500)^{-3/2} = -0.0001 < 0$. Hence, the business will realize maximum profit if 2500 units of the commodity are produced. The optimal price per unit (demand) is $p(2500) = \dfrac{50}{\sqrt{2500}} = \dfrac{50}{50} = 1$ dollar.

2. A certain company faces market demand given by $p = 48 - 3x$. This company has cost given by $C(x) = 2x^2 - 12x + 100$. Find:

   (a) Price when revenue is maximized. [hint: $R(x) = xp = 48x - 3x^2$, ans: $p = 24$]

   (b) Revenue when cost is maximized. [ans: $R(3) = 117$]

   (c) Maximum possible profit. [hint: $P(x) = -5x^2 + 60x - 100$, ans: $P(6) = 80$]

   (d) Break Even point. [hint: $R(x) = C(x)$, ans: $x = 2$ and $x = 10$]

3. A monopolist faces the demand function $p = 200 - x$. The total cost is $C = 100 - 40x + 5x^2$.

   (a) Write down the monopolist's profit as a function of the quantity produced x.

   (b) Find the profit-maximizing level of production and confirm that your solution is a maximum.

   (c) How does the profit-maximizing level of production change if the government imposes a lump sum tax, L?

   Solution

   (a) Given $x = 200 - p$, we have $p = 200 - x$. The profit function is given by

$$P(x) = xp(x) - C(x) = x(200 - x) - \left(100 - 40x + 5x^2\right) = -6x^2 + 240x - 100$$

   (b) The marginal profit (MP) is given by MP $= \dfrac{dP}{dx} = -12x + 240$. At maxima or minima, $\dfrac{dP}{dx} = 0$. That is,

$$-12x + 240 = 0 \quad \Rightarrow \quad x = 20$$

   We need to test if this value of x will lead to maximum profit. Now, $\dfrac{d^2P}{dx^2} = -12 < 0$. Hence, the business will realize maximum profit if 20 units of the commodity are produced.

(c) Since the government imposes a lump tax of L per each unit sold, the total tax bill is given by

$$T = Lx$$

The profit function, given a lump sum tax L, is given by $P = -6x^2 + 240x - 100 - T$, i.e.,

$$P = -6x^2 + 240x - 100 - Lx$$

At maximum or minimum, $\frac{dP}{dx} = 0$. That is, $-12x + 240 - L = 0$.

$$\Rightarrow \quad x = 20 - \frac{L}{12}$$

This is the optimal level of production. The government raises a total tax bill of

$$T = Lx = L\left(20 - \frac{L}{12}\right) \quad \Rightarrow \quad T = 20L - \frac{L^2}{12}$$

Differentiating with respect to L, we have $\frac{dT}{dL} = 20 - \frac{L}{6}$. To maximize the government's tax revenue, then $\frac{dT}{dL} = 0$. That is,

$$20 - \frac{L}{6} = 0 \quad \Rightarrow \quad L = 120$$

Thus, $L = 120$ maximizes the government's tax revenue? Therefore, the new profit-maximizing level of production is $x = 20 - \frac{L}{12} = 20 - \frac{120}{12} = 10$ units.

Exercise:

1. For a production level of x units of a commodity, the cost function is $C(x) = 100 + 30x$ and the demand function is $p(x) = 90 - x$. What price p will maximize profit?

   Solution

   The profit function is given by

   $$P(x) = xp(x) - C(x) = x(90 - x) - (100 + 30x) = 90x - x^2 - 100 - 30x = 60x - x^2 - 100$$

   The marginal profit (MP) is given by MP $= \frac{dP}{dx} = 60 - 2x$. At maxima or minima, $\frac{dP}{dx} = 0$. That is,

   $$60 - 2x = 0 \quad \Rightarrow \quad x = 30$$

   Now, $\frac{d^2P}{dx^2} = -2$. When $x = 30$, $P''(x) = -2 < 0$. Hence, the profit-maximizing level of production $x = 30$ units. Therefore, the optimal unit price is $p(30) = 90 - 30 = 60$ dollars.

2. A monopolist faces the demand function $x = 10 - 0.5p$. The total cost consists of a fixed overhead of 28 dollars plus production cost of 2 dollars per unit. [hint: $C(x) = 2x + 28$]

   (a) Write down the monopolist's profit as a function of the quantity produced x.

   (b) Find the profit-maximising level of production and confirm that your solution is a maximum.

   (c) Find the break-even points. What is the slope of the profit function at each of the break-even points?

   (d) Assume now that the government imposes a fixed tax of t dollars per each unit sold. What t maximises the government's tax revenue?

3. A manufacturer estimates that if x units of a particular commodity are produced, the total cost will be $C(x)$ dollars, where $C(x) = x^3 - 24x^2 + 350x + 338$.

(a) At what level of production will the marginal cost be minimized?

(b) At what level of production will the average cost be minimized?

4. A manufacturer can produce digital recorders at a cost of 50 dollars each. It is estimated that if the recorders are sold for p dollars each, consumers will by $x = 120 - p$ recorders each month.

 (a) Express the manufacturer's profit P as a function of x. [ans: $P(x) = 50x - x^2$]

 (b) What is the average rate of change in profit obtained as the level of production increases from $x = 0$ to $x = 20$ recorders. [ans: $= 30$]

 (c) At what rate is profit changing when $x = 20$ recorders are produced? Is the profit increasing or decreasing at this level of production? [ans: $P'(20) = 10$, increasing]

5.4.3 Kinematics

The motion of a particle P along a straight line is completely described by the equation $S = f(t)$, where $t > 0$ is time and S is the distance of P from a fixed point O in its path. The velocity of P at time t is $V = \dfrac{dS}{dt}$.

- ☐ If $V > 0$, P is moving in the direction of increasing S.
- ☐ If $V < 0$, P is moving in the direction of decreasing S.
- ☐ If $V = 0$, P is instantaneously at rest.

The acceleration of P at time t is $a = \dfrac{dV}{dt} = \dfrac{d^2S}{dt^2}$.

- ☐ If $a > 0$, V is increasing.
- ☐ If $a < 0$, V is decreasing.
- ☐ If V and a have the same sign, the speed of P is increasing.
- ☐ If V and a have the opposite signs, the speed of P is decreasing.

Example(s):

1. A body moves along a straight line according to the law $S = t^3 - 6t^2 + 9t + 4$. Find

 (a) S and a when $V = 0$

 (b) S and V when $a = 0$

 (c) when is S increasing?

 Solution

 (a) $\dfrac{dS}{dt} = V = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ and $a = \dfrac{dV}{dt} = 6t - 12 = 6(t-2)$. When $V = 0$, $t = 1$ or $t = 3$.

 - ☐ When $t = 1$, $S = (1)^3 - 6(1)^2 + 9(1) + 4 = 8$ and $a = 6(1-2) = -6$.
 - ☐ When $t = 3$, $S = (3)^3 - 6(3)^2 + 9(3) + 4 = 4$ and $a = 6(3-2) = 6$.

 (b) When $a = 0$, we have $6(t-2) = 0 \quad \Rightarrow \quad t = 2$. When $t = 2$, $S = (2)^3 - 6(2)^2 + 9(2) + 4 = 6$ and $V = 3(2-1)(2-3) = -3$

 (c) S is increasing when $V > 0$ i.e., when $t < 1$ and $t > 3$.

2. A body moves in a straight line so that the distance moved S metres is given in terms of the time t seconds by $S = t^3 - t^2$. Find an expression for the acceleration of the body at time t and find the times at which the body is at rest.

Solution

$\frac{dS}{dt} = V = 3t^2 - 2t$ and $\frac{d^2S}{dt^2} = \frac{dV}{dt} = a = 6t - 2$. The body is at rest when $V = 0$, i.e., $3t^2 - 2t = 0 \quad \Rightarrow \quad t = 0$ or $t = \frac{2}{3}$ seconds.

3. The distance S moved in a straight line by a particle in time t is given by $S = bt^2 + ct + d$, where b, c and d are constants. If V is the velocity of the particle at time t, show that $4b(S-d) = V^2 - c^2$.

Proof. $V = \frac{dS}{dt} = 2bt + c$. Now,

$$\begin{aligned} 4b(S - d) &= 4b(bt^2 + ct + d - d) = 4b^2t^2 + 4bct) = (2bt + c)^2 - b^2 \\ &= V^2 - c^2 \end{aligned}$$

□

Exercise:

1. The displacement S at time t of a moving particle is given by $S = b\sin 2t + c\cos 2t$, where b and c are constants. If V is the speed at time t, prove that $V = 2\sqrt{b^2 + c^2 - S^2}$.
2. A particle moves in a horizontal line according to the law $S = t^4 - 6t^3 + 12t^2 - 10t + 3$. Find
   (a) the velocity and acceleration
   (b) when the particle is at rest
3. If the velocity of a body varies inversely as the square root of the distance, prove that the acceleration varies as the fourth power of the velocity.
4. A body moves in a straight line so that its distance S metres from a fixed point O at time t seconds is given by $S = (t - 2)^2(2t - 7)$. Find when the body passes through O and the velocity and acceleration each time it passes. Find also the minimum value of the velocity.
5. The velocity V m/s of a particle which has traveled a distance S metres from a fixed point is given by $V^2 = 16S$. Find the acceleration of the particle.
6. A ball is thrown vertically upwards so that its height S metres after t seconds is given by $S = \frac{1}{27}t^2 + 4\sqrt{t}$. Find its:
   i) velocity at any time t.
   ii) acceleration when $t = 1$.
   iii) maximum height reached.

LECTURE 11

6 Introduction to integration

Integration is the reverse process of differentiation. Suppose $\dfrac{dy}{dx} = f(x)$. To obtain y, we integrate the function $f(x)$ with respect to the independent variable x. This is put in notation form as

$$y = \int f(x)dx + C,$$

where C is a constant of integration and $f(x)$ is the integrand. For example, if $\dfrac{dy}{dx} = \cos x$ then $y = \int \cos x dx = \sin x + C$. The following are some important results of integration:

(1) $\int 1dx = x + C$

(2) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, for $n \neq -1$

(3) $\int \dfrac{1}{x} dx = \ln|x| + C$

(4) $\int e^x dx = e^x + C$

(5) $\int \cos(kx)dx = \dfrac{\sin(kx)}{k} + C$, for $k \neq 0$

(6) $\int \sin(kx)dx = -\dfrac{\cos(kx)}{k} + C$, for $k \neq 0$

(7) $\int \sec^2 x dx = \tan x + C$

(8) $\int \text{cosec}^2 x dx = -\cot x + C$

(9) $\int \sec x \tan x dx = \sec x + C$

(10) $\int \text{cosec } x \cot x dx = -\text{cosec } x + C$

These types of integrals are called *indefinite* since they lack limits of integration.

6.1 Techniques of integration

6.1.1 Power rule of integration

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1},$$

where C is a constant of integration.

Example(s):

(a) $\int x^3 dx = \dfrac{x^{3+1}}{3+1} + C = \dfrac{x^4}{4} + C.$

(b) $\int x^{-7} dx = \dfrac{x^{-7+1}}{-7+1} + C = -\dfrac{x^{-6}}{6} + C.$

(c) $\int \sqrt{x} dx = \int x^{1/2} dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{2}{3}x^{3/2} + C.$

(d) $\int x(1-3x)dx = \int (x - 3x^2)dx = \dfrac{x^2}{2} - 3 \cdot \dfrac{x^3}{3} = \dfrac{x^2}{2} - x^3 + C.$

(e) $\int (2x-3)^2 dx = \int (4x^2 - 12x + 9)dx = \dfrac{4}{3}x^3 - 6x^2 + 9x + C.$

(f)

$$\begin{aligned}\int \frac{\sqrt{x} + \sqrt[3]{x} + 6}{x^2} dx &= \int \left(\frac{x^{1/2}}{x^2} + \frac{x^{1/3}}{x^2} + \frac{6}{x^2} \right) dx = \int \left(x^{-\frac{3}{2}} + x^{-\frac{5}{3}} + 6x^{-2} \right) dx \\ &= -2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{2}{3}} - 6x^{-1} + C\end{aligned}$$

(g) $\int \dfrac{x^3}{x+1} dx.$

Solution

Since the degree of the polynomial in the numerator is greater than that in the denominator, long division yields

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x+1 & x^3 \\
 & -x^3 - x^2 \\
\hline
 & -x^2 \\
 & x^2 + x \\
\hline
 & x \\
 & -x - 1 \\
\hline
 & -1
\end{array}
$$

Therefore,

$$\int \frac{x^3}{x+1} dx = \int \left(x^2 - x + 1 - \frac{1}{x+1}\right) dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$$

6.1.2 Substitution method

This technique requires that a new variable, say u, is introduced in the integrand to reduce the problem to a form in which power rule of integration can be applied.

Example(s):

(a) Evaluate $\int (2x+1)^{1/3} dx$.

Solution

Let $u = 2x + 1$. Differentiating with respect to x yields $\frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{du}{2}$. Substituting in the given integral, we get

$$\int (2x+1)^{1/3} dx = \int u^{1/3} \frac{du}{2} = \frac{1}{2} \int u^{1/3} du = \frac{1}{2}\left[\frac{3}{4} u^{4/3}\right] + C = \frac{3}{8}(2x+1)^{4/3} + C$$

(b) Evaluate $\int \frac{x}{(1-x^2)^3} dx$.

Solution

Let $u = 1 - x^2$. Differentiating with respect to x yields $\frac{du}{dx} = -2x \quad \Rightarrow \quad dx = -\frac{du}{2x}$. Substituting in the given integral, we get

$$\int \frac{x}{(1-x^2)^3} dx = \int \frac{x}{u^3}\left(-\frac{du}{2x}\right) = -\frac{1}{2} \int u^{-3} du = -\frac{1}{2}\left[\frac{1}{-2} u^{-2}\right] + C = \frac{1}{4}(1-x^2)^{-2} + C$$

(c) Evaluate $\int \sec^2(5x+1) dx$.

Solution

Let $u = 5x + 1$. Differentiating with respect to x yields $\frac{du}{dx} = 5 \quad \Rightarrow \quad dx = \frac{du}{5}$. Substituting in the given integral, we get

$$\int \sec^2(5x+1) dx = \int \sec^2(u)\left(\frac{du}{5}\right) = \frac{1}{5} \int \sec^2(u) du = \frac{1}{5} \tan u + C = \frac{1}{5} \tan(5x+1) + C$$

Exercise:

1. Evaluate the following integrals

   (a) $\int (7x-2)^3 dx$. [hint: put $u = 7x - 2$, ans: $= \frac{1}{28}(7x-2)^4 + C$]

(b) $\int 3x\sqrt{1+2x^2}dx.$ [hint: put $u = 1+2x^2$, ans: $= \frac{1}{2}(1+2x^2)^{3/2} + C$]

(c) $\int \frac{1}{(x+1)^2}dx.$ [hint: put $u = x+1$, ans: $= \frac{-1}{x+1} + C$]

(d) $\int (3x+5)^{-2}dx.$ [hint: put $u = 3x+5$, ans: $= -\frac{1}{3}(3x+5)^{-1} + C$]

(e) $\int \operatorname{cosec}^2\left(\frac{x-1}{3}\right)dx.$ [hint: put $u = \frac{x-1}{3}$, ans: $= -3\cot\left(\frac{x-1}{3}\right) + C$]

(f) $\int \frac{(\ln x)^2}{x}dx.$ [hint: put $u = \ln x$, ans: $= \frac{1}{3}(\ln x)^3 + C$]

(g) $\int x(3-5x^2)^4dx.$ [hint: put $u = 3-5x^2$, ans: $= -\frac{1}{50}(3-5x^2)^5 + C$]

(h) $\int \frac{x+1}{\sqrt[5]{x^2+2x+7}}dx.$ [hint: put $u = x^2+2x+7$, ans: $= \frac{5}{8}(x^2+2x+7)^{4/5} + C$]

(i) $\int x(2+3x)^4dx.$ [hint: put $u = 2+3x$, ans: $= \frac{1}{3}\left(\frac{(2+3x)^6}{18} - \frac{2(2+3x)^5}{15}\right) + C$]

(j) $\int x^2(1+4x^3)^3dx.$ [hint: put $u = 1+4x^3$, ans: $= \frac{1}{48}(1+4x^3)^4 + C$]